SYLVESTER'S IDENTITY

Tewodros Amdeberhan

Theorem: [**B**] Let $\mathbf{x}_n = (x_1, ..., x_n)$. Then,

$$a_k(\mathbf{x}_n) = \sum_{m=1}^{n} x_m^{n+k-1} \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{1}{x_m - x_i}, \qquad \text{where} \qquad \prod_{i=1}^{n} \frac{1}{1-x_i t} = \sum_{k=0}^{\infty} a_k(\mathbf{x}_n) t^k. \tag{1}$$

Lemma: If $P(y) := \sum_{m=1}^{n} f_m(y)$, then $P(y) \equiv 1$ where

$$f_m(y) := \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{1 - x_i y}{1 - x_i/x_m}. \tag{2}$$

Proof: Suppressing other variables, $P(y)$ and the $f_m(y)$'s are polynomials of degree $n-1$ in y. Moreover, $f_m(1/x_j) = \delta_{m,j}$ and hence $P(1/x_j) = 1$ for $j = 1, 2, \ldots, n-1$. So, $P(y)$ is a constant! $\square$

Proof of theorem: Induction on n: (1) is trivial for n=1. Application of Cauchy's product rule and induction assumption in

$$\prod_{i=1}^{n+1} \frac{1}{1-x_i t} = \frac{1}{1-x_{n+1}t} \prod_{i=1}^{n} \frac{1}{1-x_i t} = \sum_{r=0}^{\infty} a_r(x_{n+1}) t^r \sum_{s=0}^{\infty} a_s(\mathbf{x}_n) t^s \tag{3}$$

produces the coefficient $a_k(\mathbf{x}_{n+1})$ of t^k as

$$\begin{aligned}
\sum_{s=0}^{k} x_{n+1}^{k-s} \sum_{m=1}^{n} x_m^s \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{1}{1-x_i/x_m} &= \sum_{m=1}^{n} x_{n+1}^k \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{1}{1-x_i/x_m} \sum_{s=0}^{k} x_m^s x_{n+1}^{-s} \\
&= \sum_{m=1}^{n} x_{n+1}^k \left(\frac{x_m^{k+1} - x_{n+1}^{n+1}}{x_m - x_{n+1}} \right) \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{1}{1-x_i/x_m} \\
&= \sum_{m=1}^{n} x_m^k \prod_{\substack{i=1\\ i\neq m}}^{n+1} \frac{1}{1-x_i/x_m} - \sum_{m=1}^{n} \frac{x_{n+1}^k}{x_m/x_{n+1} - 1} \prod_{\substack{i=1\\ i\neq m}}^{n} \frac{1}{1-x_i/x_m}. \qquad (4)
\end{aligned}$$

Now use of the above Lemma (with $y = 1/x_{n+1}$) in the second sum of (4) completes the proof. $\square$

Reference:

[**B**] G. Bhatnagar, *A short proof of an identity of Sylvester,* unpublished.

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