Heinig's conjectured condition (S)

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Abstract

1 Introduction

Mackey, Mackey and Petrovic [2] posed the question *"Are all matrices similar to a Toeplitz matrix?"* They showed that every $n \times n$ complex nonderogatory matrix is similar to a unique upper Hessenberg Toeplitz matrix, and also that every lower dimensional ($n \leq 4$) complex matrix is similar to a Toeplitz matrix. The general case is treated in an important paper [1] where Heinig gave the answer in the negative. Among the many results spread about the latter article, we have

Theorem 6.1. *Assume $m \geq 4$ is such that condition (S) is fulfilled. Then the class $\mathscr{M}(2m, m, m-1)$ is empty. That means there is no Toeplitz matrix that is similar to*

$$\left(\bigoplus_{j=1}^{m-1} S_2\right) \oplus 0 \oplus c \qquad \textit{or} \qquad S_3 \oplus \left(\bigoplus_{j=1}^{m-2} S_2\right) \oplus 0 \qquad \textit{if } c \neq 0. \tag{1}$$

First some nomenclature is in order. Define the polynomials $p_j(t)$ recursively by (**please note that we included $p_0(t) = 1$ which makes no difference towards the claim (S)**)

$$p_0(t) := 1 = p_1(t), p_2(t) = t, \qquad p_j(t) = -\frac{1}{2}\sum_{k=1}^{j-1} p_k(t)p_{j-k}(t), \qquad j \geq 3. \tag{2}$$

The generating function $p(z,t) = \sum_{j=0}^{\infty} p_j(t) z^j$ is given by (**please observe the precious "mistake" - there should not have been -1**)

$$p(z,t) = \left(1 + 2z + z^2(1+2t)\right)^{1/2}. \tag{3}$$

from which the following expression for the $p_j(t)$ follows: (*typo in the paper [1] is corrected here*)

$$p_j(t) = \sum_{k=0}^{j/2} 2^{j-k} \binom{1/2}{j-k} \binom{j-k}{k} \left(t + \frac{1}{2}\right)^k.$$

Condition (S) ([1], page 528) *For $m \geq 4$, the system of $m-2$ equations*

$$p_{m+2}(t) = p_{m+3}(t) = \cdots = p_{2m-1}(t) = 0$$

has only the trivial solution $t = 0$.

Then Heinig further remarked that *"This is true for $m = 4$ and $m = 5$. We conjecture that it is valid for all m, since it seems that even two consecutive polynomials in the sequence $\{p_k(t)\}$ are coprime."*

Our objective at present is to dispense with the assumption (S) instead by proving it, as conjectured by Heinig. Therefore this completes the negative answer to the inverse Jordan structure problem in the even dimensional case: *for $n \geq 4$ there exist $n \times n$ matrices which are not similar to a Toeplitz matrix.*

2 The Recurrence

Strengthened Condition (S') *For $m \geq 2$, $p_{m+1}(t) = p_m(t) = 0$ has only the trivial solution $t = 0$.*

Proof The sequence $\{p_m(t)\}_m$ also satisfies another recurrence (**with initial conditions** $p_0(t) = p_1(t) = 1$)

$$(4) \qquad (m+2)p_{m+2}(t) + (2m+1)p_{m+1}(t) + (m-1)(2t+1)p_m(t) = 0.$$

Let $F(j,k) := 2^{j-k}\binom{1/2}{j-k}\binom{j-k}{k}\left(t+\frac{1}{2}\right)^k$, and $G(j,k) := -2\frac{(j-1)(2j-2k-1)k}{(j+1-2k)(j+2-2k)}F(j,k)$ then one can check, preferably using a symbolic software, that

$$(j+2)F(j+2,k)+(2j+1)F(j+1,k)+2(j-1)(t+\frac{1}{2})F(j,k) = G(j,k+1)-G(j,k).$$

Telescoping: Sum over all $-\infty < k < \infty$ and also notice that

$$\sum_{k=-\infty}^{\infty} F(r,k) = \sum_{k=0}^{r/2} F(r,k) = p_r(t) \qquad \text{while} \sum_{k=-\infty}^{\infty} G(j,k+1) = \sum_{k=-\infty}^{\infty} G(j,k),$$

since $G(j,k)$ has compact support, the assertion (4) follows. [An alternative route might be transforming the recursive formula (4) above into a differential equation (w.r.t. $\frac{d}{dz}$) and check that the generating function $p(z,t)$ satisfies it.] At any rate, condition (S') can now be proven by induction on m with the added observation that none of the first three terms $p_2 = t, p_3 = -t, p_4 = t - \frac{1}{2}t^2$ (and hence all the rest) vanishes at $t = -\frac{1}{2}$. $\square$

We will give an alternating proof of (4).

Lemma 1 *The sequence* $\{p_j(t)\}_{j=0}^{\infty}$ *with* $p_0(t) = p_1(t) = 1$ *and*

$$(5) \qquad (m+2)p_{m+2}(t) + (2m+1)p_{m+1}(t) + (m-1)(2t+1)p_m(t) = 0$$

has a generating function given by $h(z,t) = \left(1 + 2z + z^2(1+2t)\right)^{1/2}$.

Proof Let $h(z,t) = \sum_{m=0}^{\infty} p_m(t) z^m$, then $\frac{d}{dz}h(z,t) = \sum_{m=1}^{\infty} m p_m(t) z^{m-1}$. Multiply (5) throughout by z^{m+1} and sum over all $0 \le m < \infty$ to get

$$\sum_{m\ge 0}(m+2)p_{m+2}(t)z^{m+1} + \sum_{m\ge 0}(2m+1)p_{m+1}(t)z^{m+1} + \sum_{m\ge 0}(m-1)(2t+1)p_m(t)z^{m+1} = 0.$$

$$(6) \qquad \sum_{m\ge 0}(m+2)p_{m+2}(t)z^{m+1} + \sum_{m\ge 0}(2m+2)p_{m+1}(t)z^{m+1}$$
$$- \sum_{m\ge 0} p_{m+1}(t)z^{m+1} + (2t+1)\sum_{m\ge 0} m p_m(t) z^{m+1} - (2t+1)\sum_{m\ge 0} p_m(t) z^{m+1} = 0.$$

$$(7) \qquad \sum_{m\ge 2} m p_m(t) z^{m-1} + 2z \sum_{m\ge 1} m p_m(t) z^{m-1} - \sum_{m\ge 1} p_m(t) z^m$$
$$+ z^2(2t+1)\sum_{m\ge 1} m p_m(t) z^{m-1} - z(2t+1)\sum_{m\ge 0} p_m(t) z^m = 0.$$

$$\left(\frac{dh}{dz}(z,t) - p_1\right) + 2z\frac{dh}{dz}(z,t) - (h(z,t) - p_0) + z^2(2t+1)\frac{dh}{dz}(z,t) - z(2t+1)h(z,t) = 0.$$

$$(1 + 2z + z^2(2t+1))\frac{dh}{dz}(z,t) - (1 + z(2t+1))h(z,t) - p_1 + p_0 = 0.$$

$$(1 + 2z + z^2(2t+1))\frac{dh}{dz}(z,t) = (1 + z(2t+1))h(z,t), \qquad 1 = p_0(t) = h(0,t).$$

The last separable initial-value ODE produces $h(z,t) = \left(1 + 2z + z^2(1+2t)\right)^{1/2}$. This completes the proof and reconstruction of $p(z,t)$ from (3) above. □

3 Acknowledgments

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References

[1] Heinig G *Not every matrix is similar to a Toeplitz matrix* Linear Algebra Appl. 332-334 (2001) pp. 519-531.

[2] Mackey D S, Mackey N, Petrovic S *Is every matrix similar to a Toeplitz matrix?* Linear Algebra Appli. 297 (1999) pp. 87-105.