An identity millions of times less useful than the sine addition formula

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Abstract

Bill Gosper introduced a six-factor trigonometric identity that he called "millions of times less useful than the sine addition formula." Yet this curious identity admits proofs using elementary trigonometry, integration, and residues, and is closely related to Ptolemy's trigonometric identity. We explore its elliptic and theta-function counterparts through classical identities of Jacobi and Fay, and show how multiplicative telescoping illuminates several of these connections. Finally, Schur's Pfaffian identity yields a generalization to an arbitrary even number of variables. The resulting connections link elementary trigonometry and geometry with elliptic functions, complex analysis, and combinatorics.

Keywords: Trigonometric identities, Ptolemy identity, Jacobi identity, elliptic functions, Fay trisecant identity.

1 Introduction: a trigonometric identity.

In a celebrated article [2], W. Gosper qualifies the identity

$$\begin{aligned}
&\sin(x-w)\sin(x-y)\sin(y-w)\sin(y-z)\sin(z-w)\sin(z-x)\\
&+\cos(x-w)\cos(x-y)\sin(y-w)\cos(y-z)\cos(z-w)\sin(z-x)\\
&+\cos(x-w)\sin(x-y)\cos(y-w)\cos(y-z)\sin(z-w)\cos(z-x)\\
&+\sin(x-w)\cos(x-y)\cos(y-w)\sin(y-z)\cos(z-w)\cos(z-x)=0 \quad (1)
\end{aligned}$$

as "just like the factorization of x^2-y^2, or the sine addition formula, only millions of times less useful."

Our goal here is to produce multiple approaches of this identity, each using tools borrowed from a different field of mathematics such as trigonometry, combinatorics, real analysis, complex analysis and elliptic functions. These different approaches will gradually reveal the connections between Gosper's identity and more profound results. Some of them will also allow us to derive an extension of Gosper's identity to a new version with an arbitrary even number of variables.

For instance, there is a brute force approach. First, denote the left-hand side of (1) by G (for Gosper). Next, use trig identities to linearize products of

the 6 trigonometric functions each producing 12 terms for a total of 48 terms. Omitting details, one obtains $8G =$

$$\begin{aligned}
&\cos(x_1+x_2+x_3-X_1-X_2-X_3)-\cos(x_1+x_2+x_3+X_1+X_2+X_3)\\
&+\cos(x_1-x_2+x_3-X_1+X_2-X_3)-\cos(x_1-x_2+x_3+X_1-X_2+X_3)\\
&+\cos(x_1+x_2-x_3-X_1-X_2+X_3)-\cos(x_1+x_2-x_3+X_1+X_2-X_3)\\
&+\cos(x_1-x_2-x_3-X_1+X_2+X_3)-\cos(x_1-x_2-x_3+X_1-X_2-X_3),
\end{aligned}$$

where we let $x_1 = x-w, x_2 = x-y, x_3 = y-w, X_1 = y-z, X_2 = z-w$ and $X_3 = z-x$. Since $x_1 = x_2+x_3, x_3 = X_1+X_2, X_2 = X_3+x_1$, it follows that $8G = 0$ proves Gosper identity (1). The brute-force calculation establishes the identity, but its extensive cancellations invite a more structured elementary proof.

2 An elementary route.

Is there a simpler relation that explains the six-factor expression? In this section, a symmetry-based proof reduces the computation; Ptolemy's identity offers a geometric explanation for the same cancellation. The shifted Ptolemy relations give one route to Gosper's identity, while a change of notation reveals an even shorter algebraic route.

2.1 Trigonometry and symmetry.

We look now for a shorter and less painful proof of Gosper's identity (1) that still relies on trigonometry alone. The idea is to exploit symmetry in order to lower the complexity of the computations.

We start by remarking that Gosper's identity is not symmetric: some terms seem to be missing. More precisely, let us define, with $1 \le i \le 3$ and the variables $\{x_i, X_i\}$ from Section 1,

$$p_i = \cos x_i \cos X_i,\ q_i = \sin x_i \sin X_i,$$

so that Gosper's identity is $G = q_1q_2q_3 + p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3 = 0$. This expression seems to be part of the larger symmetric expression

$$\begin{aligned}
P_+ &= p_1p_2p_3 + q_1p_2p_3 + p_1q_2p_3 + p_1p_2q_3 + q_1q_2p_3 + q_1p_2q_3 + p_1q_2q_3 + q_1q_2q_3\\
&= (p_1+q_1)(p_2+q_2)(p_3+q_3),
\end{aligned}$$

except that some of the terms in P_+ do not appear in G: we notice that these missing terms are exactly the terms that appear with a positive sign in the expansion

$$\begin{aligned}
P_- &= p_1p_2p_3 - q_1p_2p_3 - p_1q_2p_3 - p_1p_2q_3 + q_1q_2p_3 + q_1p_2q_3 + p_1q_2q_3 - q_1q_2q_3\\
&= (p_1-q_1)(p_2-q_2)(p_3-q_3).
\end{aligned}$$

We deduce that the difference

$$P_+ - P_- = (p_1+q_1)(p_2+q_2)(p_3+q_3) - (p_1-q_1)(p_2-q_2)(p_3-q_3)$$

coincides with $2G$. Since

$$\begin{aligned} p_i + q_i &= \cos x_i \cos X_i + \sin x_i \sin X_i = \cos(x_i - X_i) \qquad \text{and} \\ p_i - q_i &= \cos x_i \cos X_i - \sin x_i \sin X_i = \cos(x_i + X_i), \end{aligned}$$

we infer that

$$2G = \prod_{j=1}^{3} \cos(x_j - X_j) - \prod_{j=1}^{3} \cos(x_j + X_j).$$

With the definitions of $\{x_i, X_i\}$, it is easily checked that $G = 0$.

2.2 From Ptolemy to Gosper: A linear operator approach.

Ptolemy's identity is familiar to geometers: it states that, for a quadrilateral whose vertices lie on the unit circle, the product of the lengths of its diagonals coincides with the sum of the products of the lengths of the pairs of opposite sides.

We wonder now if a geometer equipped with the knowledge of Ptolemy's identity

$$\sin(t-z)\sin(x-y) - \sin(t-y)\sin(x-z) + \sin(t-x)\sin(y-z) = 0 \qquad (2)$$

and elementary linear operators could recover Gosper's identity. Define the variables $p = \sin(x-w)\sin(y-z)$, $q = \sin(x-y)\sin(z-w)$, $r = \sin(y-w)\sin(z-x)$ so that Ptolemy's identity (2) reads

$$p + q + r = 0. \qquad (3)$$

Define, moreover, $P = \cos(x-w)\cos(y-z)$, $Q = \cos(x-y)\cos(z-w)$ and $R = \cos(y-w)\cos(z-x)$ so that Gosper's identity is

$$pqr + PQr + PqR + pQR = 0.$$

Introduce the map $A : (x, y, z, w) \mapsto (x + \pi/2, y, z + \pi/2, w + \pi)$ leaving y fixed, and the map $B : (x, y, z, w) \mapsto (x + \pi/2, y + \pi/2, z, w + \pi)$ leaving z fixed, so that A and B act on p, q, r, respectively, as

$$Ap = P, Aq = -Q, Ar = -r \qquad \text{and} \qquad Bp = -P, Bq = -q, Br = R.$$

Ptolemy's equation and its two shifted versions obtained by application of mappings A and B, are

$$\begin{cases} p + q + r & = 0, \\ P - Q - r & = 0, \\ -P - q + R & = 0, \end{cases}$$

so that $pq + PQ = pq + P(p + q + P) = (P + q)(P + p) = R(Q - q)$ and

$$\begin{aligned} pQ + qP &= pQ + q(Q + r) = pQ + q(-p + Q - q) \\ &= (p + q)(Q - q) = -r(Q - q). \end{aligned}$$

We deduce the left-hand side of Gosper's identity as

$$\begin{aligned} pqr + PQr + PqR + pQR &= r(pq + PQ) + R(pQ + Pq) \\ &= rR(Q - q) + R(-r(Q - q)) = 0, \end{aligned}$$

which produces a proof of Gosper's identity.

In conclusion, Gosper's identity can be obtained by combining three equivalent versions of Ptolemy's identity. We notice how little trigonometry is required in this approach, as most of it is already encoded in Ptolemy's identity.

2.3 Another trigonometric proof.

We invoke Ptolemy's identity in the form

$$\sin(w - x)\sin(y - z) + \sin(w - y)\sin(z - x) + \sin(w - z)\sin(x - y) = 0$$

to suggest yet another trigonometric proof of Gosper's identity: define the following

$$\begin{aligned} A &= \sin(x - w)\sin(y - z), & a &= \cos(x - y)\cos(z - w), \\ B &= \sin(x - y)\sin(z - w), & b &= \cos(x - z)\cos(y - w), \\ C &= \sin(y - w)\sin(z - x), & c &= \cos(x - w)\cos(y - z). \end{aligned}$$

Notice that elementary trigonometry produces $A = a - b,\ B = b - c,\ C = c - a$. Ptolemy's identity is easily reformulated as

$$A + B + C = 0.$$

Gosper's identity,

$$ABC + Aab + Bbc + Cca = 0,$$

is now checked as follows: start from the elementary identity

$$(u - v)(v - w)(w - u) + (w - u)uw + (v - w)vw + (u - v)uv = 0 \quad (4)$$

and substitute $u = a, v = b, w = c$.

3 Orthogonality.

Having proved Gosper's identity through trigonometry and geometry, we turn to orthogonality for an analytic perspective. Denote as

$$f(x, y, z, w) = \sum_{i=1}^{4} f_i(x, y, z, w)$$

the left-hand side of (1), and consider the integral

$$I = \langle f, f \rangle = \int_{[0,2\pi]^4} f^2 \left(x, y, z, w\right) dxdydzdw.$$

Expanding the square produces

$$I = \sum_{i=1}^{4} \langle f_i, f_i \rangle + \sum_{1 \le i \ne j \le 4} \langle f_i, f_j \rangle.$$

By symmetry, we have $\langle f_i, f_i \rangle = \langle f_1, f_1 \rangle,\ i = 2, 3, 4$ and $\langle f_i, f_j \rangle = \langle f_3, f_4 \rangle, i \ne j$.

Our goal is to derive an analytic proof for Gosper's identity by showing that the integral I vanishes. This naive approach is instantly rewarded by an unexpected result: the diagonal part $\langle f_1, f_1 \rangle$ of integral I, which equals to (for notational simplicity write $\Psi(t) = \sin^2 t, d\Omega = dxdydzdw$ and $\square = [0, 2\pi]^4$)

$$\int_{\square} \Psi\left(x - w\right) \Psi\left(x - y\right) \Psi\left(y - w\right) \Psi\left(y - z\right) \Psi\left(z - w\right) \Psi\left(z - x\right) d\Omega,$$

is identified as the famous Dyson Vandermonde integral [3]

$$\int_{[0,2\pi]^n} \prod_{1 \le j < k \le n} \sin^2\left(u_j - u_k\right) \prod_{i=1}^{n} du_i = \frac{n!\left(2\pi\right)^n}{4^{\frac{n(n-1)}{2}}}$$

in the case $n = 4$, so that

$$\langle f_1, f_1 \rangle = \frac{4!\left(2\pi\right)^4}{4^6} = \frac{3}{32}\pi^4.$$

We now compute the non-diagonal part $\langle f_3, f_4 \rangle$, noticing that $16\langle f_3, f_4 \rangle$ is the integral (again, for simplicity let $\Phi(t) = \sin(t)$)

$$J = \int_{\square} \Phi\left(2x - 2w\right) \Phi\left(2x - 2y\right) \Phi^2\left(y - w\right) \Phi\left(2y - 2z\right) \Phi\left(2z - 2w\right) \Phi^2\left(z - x\right) d\Omega.$$

Remarkably, the computation of this quadruple integral is based solely on the simple evaluation

$$I(a, b, c) = \int_0^{2\pi} \sin(2a - 2t) \sin^2(b - t) \sin(2c - 2t)\, dt = \frac{\pi}{2} \cos(2a - 2c).$$

The integral over w in J is identified as $I(x, y, z) = \frac{\pi}{2} \cos(2x - 2z)$ so that

$$J = \frac{\pi}{2} \int_{[0,2\pi]^3} \cos\left(2x - 2z\right) \sin\left(2x - 2y\right) \sin\left(2y - 2z\right) \sin^2\left(z - x\right) dxdydz.$$

Next, using $\cos(2x - 2z) = \sin\left(2\left(z + \frac{\pi}{4}\right) - 2x\right)$, the integral over x in J is identified as $-I(y, z, z + \frac{\pi}{4}) = -\frac{\pi}{2} \sin\left(2y - 2z\right)$ so that

$$J = -\frac{\pi^2}{4} \int_{[0,2\pi]^2} \sin^2\left(2y - 2z\right) dydz.$$

In addition, since

$$I(a,b,c) + I(a,b+\frac{\pi}{2},c) = \int_0^{2\pi} \sin(2a-2t)\sin(2c-2t)\,dt,$$

the integral over z in J is identified as $I(y,b,y) + I(y,b+\frac{\pi}{2},y) = \pi$ and finally,

$$J = -\frac{\pi^4}{2},\ \langle f_3, f_4\rangle = -\frac{\pi^4}{32}.$$

Counting the number of identical diagonal terms (4) and non-diagonal terms (12) in the integral I allows us to deduce the final result

$$I = 4\langle f_1, f_1\rangle + 12\langle f_3, f_4\rangle = 4\left(\frac{3}{32}\pi^4\right) + 12\left(-\frac{\pi^4}{32}\right) = 0,$$

which proves Gosper's identity.

4 Differentiation: an invariant in disguise.

We wonder if differentiation will produce a shorter and easier proof than the previous integration-based one: denote $a = x-w, b = y-w, c = z-w$. Then $x-y = a-b$, $y-z = b-c$, and $z-x = c-a$.

A preliminary trigonometric step simplifies the process: applying the standard product-to-sum formulas to the four triple products involving a in f gives:

$$f(x,y,z,w) = \frac{1}{4}\sin 2(b-c)\sin(b+c-a) + \frac{1}{4}\sin 2c\sin(c-b-a) + \frac{1}{4}\sin 2b\sin(a-b+c).$$

Because $\partial a/\partial x = 1$, differentiation yields

$$\frac{\partial f}{\partial x} = -\frac{1}{4}\sin 2(b-c)\cos(b+c-a) - \frac{1}{4}\sin 2c\cos(c-b-a) + \frac{1}{4}\sin 2b\cos(a-b+c).$$

From the linearizations, we get

$$\begin{aligned}
\sin 2(b-c)\cos(b+c-a) &= \frac{1}{2}\left[\sin(3b-c-a) + \sin(a+b-3c)\right],\\
\sin 2c\cos(c-b-a) &= \frac{1}{2}\left[\sin(3c-b-a) + \sin(a+b+c)\right],\\
\sin 2b\cos(a-b+c) &= \frac{1}{2}\left[\sin(3b-a-c) + \sin(a+b+c)\right];
\end{aligned}$$

we deduce $\frac{\partial f}{\partial x} = 0$. Hence, f is constant as a function of x. Evaluating f at $x = w$ gives Gosper's identity $f = 0$.

5 Residues and the elementary source of Ptolemy.

We wonder if a totally new approach based on complex analysis would produce Gosper's identity. The differentiation argument from above returns us to elementary trigonometry, whereas residues provide a different source for both Ptolemy's and Gosper's identities. Let us start from the observation that the elementary identity

$$(w-x)(y-z)+(w-y)(z-x)+(w-z)(x-y)=0 \tag{5}$$

can be seen as the result of Cauchy's residue theorem

$$\sum_{t\in\{x,y,z\}} res_t R(t)=0$$

applied to the rational function

$$R_1(t)=\frac{t-w}{(t-x)(t-y)(t-z)},$$

while the equally elementary identity (4)

$$(u-v)(v-w)(w-u)+(w-u)uw+(v-w)vw+(u-v)uv=0$$

can be obtained by applying Cauchy residue theorem to the rational function

$$R_2(t)=\frac{t^2}{(t-u)(t-v)(t-w)}.$$

We derive the Ptolemy identity from identity (5), substituting

$$x\to\cot x, y\to\cot y, z\to\cot z, w\to\cot w \tag{6}$$

and then simplifying by the addition law for the cotangent function. This suggests the following shortcut.

Theorem 1. *The application of Cauchy residue theorem to the function*

$$f_P(t)=\frac{\sin(t-w)}{\sin(t-x)\sin(t-y)\sin(t-z)}$$

produces Ptolemy's identity, whereas the choice

$$f_G(t)=\cot(t-x)\cot(t-y)\cot(t-z)\cot(t-w)$$

produces Gosper's identity.

Proof. In the Ptolemy case, the residues are

$$\begin{aligned}
\mathrm{Res}(f_P,x)&=\lim_{t\to x}(t-x)f_P(t)=\frac{\sin(w-x)}{\sin(x-y)\sin(z-x)},\\
\mathrm{Res}(f_P,y)&=\lim_{t\to y}(t-y)f_P(t)=\frac{\sin(w-y)}{\sin(y-z)\sin(x-y)},\\
\mathrm{Res}(f_P,z)&=\lim_{t\to z}(t-z)f_P(t)=\frac{\sin(w-z)}{\sin(z-x)\sin(y-z)}.
\end{aligned}$$

Let Γ_M be the rectangular contour bounding the region $[t_0, t_0 + \pi] \times [-M, M]$, traversed counterclockwise. Choose $t_0 \in \mathbb{R}$ such that no poles lie on the vertical boundary lines $\mathrm{Re}(t) = t_0$ and $\mathrm{Re}(t) = t_0 + \pi$. After checking that the integrals along the vertical segments cancel identically for all $M > 0$:

$$\int_{t_0+\pi-iM}^{t_0+\pi+iM} f_P(t)\,dt + \int_{t_0+iM}^{t_0-iM} f_P(t)\,dt = 0,$$

and that the top and bottom line integrals are bounded by $O(e^{-2M})$ and thus tend to 0 with M, application of Cauchy's residue theorem produces

$$\lim_{M\to\infty} \frac{1}{2\pi i} \oint_{\Gamma_M} f_P(t)\,dt = 0 \implies \sum_{p\in\{x,y,z\}} \mathrm{Res}(f_P, p) = 0.$$

Substituting the residue values produces Ptolemy's identity.

We now consider

$$f_G(t) = \cot(t-x)\cot(t-y)\cot(t-z)\cot(t-w).$$

The poles are simple and occur at x, y, z, w modulo π. Let's make good use of $\lim_{t\to a}(t-a)\cot(t-a) = 1$, to compute the residues

$$\begin{aligned}
\mathrm{Res}(f_G, x) &= \cot(x-y)\cot(x-z)\cot(x-w),\\
\mathrm{Res}(f_G, y) &= -\cot(x-y)\cot(y-z)\cot(y-w),\\
\mathrm{Res}(f_G, z) &= \cot(x-z)\cot(y-z)\cot(z-w),\\
\mathrm{Res}(f_G, w) &= -\cot(x-w)\cot(y-w)\cot(z-w).
\end{aligned}$$

We apply the residue theorem to the same rectangular contour Γ_M. The vertical integrals again cancel, because f_G is π-periodic. To handle the horizontal sides, we use

$$\cot(t-a) = \begin{cases} -i + O(e^{-2\,\mathrm{Im}\,t}), & \mathrm{Im}\,t \to +\infty,\\ i + O(e^{2\,\mathrm{Im}\,t}), & \mathrm{Im}\,t \to -\infty, \end{cases}$$

uniformly in $\mathrm{Re}\,t$, for each real a. It follows that $f_G(t) = 1 + O(e^{-2|\,\mathrm{Im}\,t|})$ as $|\,\mathrm{Im}\,t| \to \infty$. Thus, the integral over the bottom side tends to π, whereas the integral over the top side tends to $-\pi$, owing to the opposite orientations. Their sum therefore tends to zero. Hence,

$$\lim_{M\to\infty} \oint_{\Gamma_M} f_G(t)\,dt = 0.$$

The residue theorem now gives

$$\sum_{p\in\{x,y,z,w\}} \mathrm{Res}(f_G, p) = 0.$$

Substituting the four residues, we obtain

$$\begin{aligned}
&\cot(x-y)\cot(x-z)\cot(x-w) - \cot(x-y)\cot(y-z)\cot(y-w)\\
&+ \cot(x-z)\cot(y-z)\cot(z-w) - \cot(x-w)\cot(y-w)\cot(z-w) = 0.
\end{aligned}$$

This is the cotangent form of Gosper's identity. Applying the cotangent addition formula and simplifying yields the desired identity.

□

6 Multiplicative telescoping and Jacobi's identity.

Residues have established additive relations for us; a ratio formulation will expose cancellations that extend to elliptic functions. Before exploring that extension, the tangent form gives one last short trigonometric proof.

6.1 An even shorter trigonometric proof.

Transforming Gosper's expression only slightly will produce an even shorter proof: divide both sides of (1) by the product

$$\cos(x-w)\cos(x-y)\cos(y-w)\cos(y-z)\cos(z-w)\cos(z-x)$$

to obtain the tangent form of Gosper's identity

$$\begin{aligned}&\tan(x-w)\tan(x-y)\tan(y-w)\tan(y-z)\tan(z-w)\tan(z-x)\\&+\tan(y-w)\tan(z-x)+\tan(x-y)\tan(z-w)\\&+\tan(x-w)\tan(y-z)=0.\end{aligned}\tag{7}$$

Now, revive one of Gosper's favorite tangent [1] identity [8, p.4]

$$\tan(x-w)\tan(w-y)\tan(y-x)=\tan(x-w)+\tan(w-y)+\tan(y-x)$$

and apply it repeatedly to the left-hand side (7). The routine cancellation proves (7) independently of our earlier justifications of (1). The details are omitted.

6.2 Multiplicative Telescoping.

The tangent calculation suggests replacing sums by products, where matching factors can cancel through multiplicative telescoping. Telescoping appears often in Gosper's experiments: let us mention for example [8, p.12]

$$\prod_{n\geq 1}\left(2\cos\left(\frac{\theta}{2^n}\right)-1\right)=\frac{1+2\cos\theta}{3}$$

as an elementary consequence of the identity

$$2\cos\theta-1=\frac{1+2\cos 2\theta}{1+2\cos\theta}.$$

[1] equivalently: if $a+b+c=0$, then $\tan a\tan b\tan c=\tan a+\tan b+\tan c$

Define the bivariate function

$$f(x,y)=\frac{1-\tanh x\tanh y}{1+\tanh x\tanh y}$$

and notice that, due to the addition formula for the hyperbolic tangent function

$$\tanh(x+y)=\frac{\tanh x+\tanh y}{1+\tanh x\tanh y},$$

f simplifies to

$$f(x,y)=\frac{\cosh(x-y)}{\cosh(x+y)}$$

so that the choice $h(z)=\cosh z$ leads to

$$\frac{1-\tanh(w-x)\tanh(y-z)}{1+\tanh(w-x)\tanh(y-z)}\frac{1-\tanh(w-y)\tanh(z-x)}{1+\tanh(w-y)\tanh(z-x)}\frac{1-\tanh(w-z)\tanh(x-y)}{1+\tanh(w-z)\tanh(x-y)}$$

$$=\frac{h(w-x-y+z)}{h(w-x+y-z)}\frac{h(w-y+x-z)}{h(w-y+z-x)}\frac{h(w-z-x+y)}{h(w-z+x-y)}=1,$$

which produces, after simplification, an immediate proof of the hyperbolic tangent version (7) of Gosper's identity. This approach extends to the elliptic case: choosing a value of the elliptic modulus k and writing $\operatorname{sn}(x)$ for $\operatorname{sn}(x,k)$ and similarly for the other elliptic functions, the function

$$f(x,y)=\frac{1-k\operatorname{sn}x\operatorname{sn}y}{1+k\operatorname{sn}x\operatorname{sn}y}$$

satisfies

$$f(x,y)=\frac{\operatorname{dn}(x-y)-k\operatorname{cn}(x-y)}{\operatorname{dn}(x+y)-k\operatorname{cn}(x+y)}=\frac{1-k\operatorname{cd}(x-y)}{1-k\operatorname{cd}(x+y)}$$

so that, with $h(z)=\operatorname{dn}(z)-k\operatorname{cn}(z)$, we obtain

$$\frac{1-k\operatorname{sn}(w-x)\operatorname{sn}(y-z)}{1+k\operatorname{sn}(w-x)\operatorname{sn}(y-z)}\frac{1-k\operatorname{sn}(w-y)\operatorname{sn}(z-x)}{1+k\operatorname{sn}(w-y)\operatorname{sn}(z-x)}\frac{1-k\operatorname{sn}(w-z)\operatorname{sn}(x-y)}{1+k\operatorname{sn}(w-z)\operatorname{sn}(x-y)}$$

$$=\frac{h(w-x-y+z)}{h(w-x+y-z)}\frac{h(w-y+x-z)}{h(w-y+z-x)}\frac{h(w-z-x+y)}{h(w-z+x-y)}=1. \tag{8}$$

Notice that this identity extends to any function $f(x,y)$ that can be expressed as

$$f(x,y)=\frac{h(x-y)}{h(x+y)}.$$

Let us state this general but elementary result:

Proposition. *If there exists a function h such that*

$$f(x,y)=\frac{h(x-y)}{h(x+y)}$$

then f satisfies $f(w-x,y-z)\,f(w-y,z-x)\,f(w-z,x-y)=1$.

Example 2. Consider $h(z) = \operatorname{cn}(z) + \imath \operatorname{sn}(z) = e^{\imath \operatorname{am}(z)}$ with $\operatorname{am}(z)$ denoting the elliptic amplitude function, so that we consider the function

$$f(x,y) = \frac{\operatorname{cn}(y) - \imath \operatorname{sn}(y) \operatorname{dn}(x)}{\operatorname{cn}(y) + \imath \operatorname{sn}(y) \operatorname{dn}(x)} = \frac{1 - \imath \operatorname{sc}(y) \operatorname{dn}(x)}{1 + \imath \operatorname{sc}(y) \operatorname{dn}(x)} = \frac{h(x-y)}{h(x+y)}.$$

Using multiplicative telescoping, we deduce

$$\frac{1 - \imath \operatorname{dn}(w-x) \operatorname{sc}(y-z)}{1 + \imath \operatorname{dn}(w-x) \operatorname{sc}(y-z)} \frac{1 - \imath \operatorname{dn}(w-y) \operatorname{sc}(z-x)}{1 + \imath \operatorname{dn}(w-y) \operatorname{sc}(z-x)} \frac{1 - \imath \operatorname{dn}(w-z) \operatorname{sc}(x-y)}{1 + \imath \operatorname{dn}(w-z) \operatorname{sc}(x-y)} = 1$$

and, after simplification,

$$\begin{aligned} &\operatorname{dn}(w-x) \operatorname{sc}(y-z) + \operatorname{dn}(w-y) \operatorname{sc}(z-x) + \operatorname{dn}(w-z) \operatorname{sc}(x-y) \\ &= \operatorname{dn}(w-x) \operatorname{sc}(y-z) \operatorname{dn}(w-y) \operatorname{sc}(z-x) \operatorname{dn}(w-z) \operatorname{sc}(x-y). \end{aligned} \tag{9}$$

Since $\operatorname{dn}(z,0) = 1$ and $\operatorname{sc}(z,0) = \tan(z)$, the limit case $k \to 0$ of identity (9) is Gosper's favorite tangent identity

$$\tan(y-z) + \tan(z-x) + \tan(x-y) = \tan(y-z) \tan(z-x) \tan(x-y).$$

Since $\operatorname{dn}(z,1) = \operatorname{sech}(z)$ and [2] $\operatorname{sc}(z,1) = \sinh(z)$, the limit case $k \to 1$ of identity (9) is

$$\begin{aligned} &\cosh(w-x) \cosh(w-y) \sinh(x-y) + \cosh(w-x) \cosh(w-z) \sinh(z-x) \\ &+ \cosh(w-y) \cosh(w-z) \sinh(y-z) = \sinh(x-y) \sinh(z-x) \sinh(y-z) \end{aligned}$$

Example 3. Choose $h(z) = \operatorname{dn}(z)$ so that

$$\begin{aligned} f(x,y) &= \frac{\operatorname{dn}(x) \operatorname{dn}(y) + k^2 \operatorname{sn}(x) \operatorname{sn}(y) \operatorname{cn}(x) \operatorname{cn}(y)}{\operatorname{dn}(x) \operatorname{dn}(y) - k^2 \operatorname{sn}(x) \operatorname{sn}(y) \operatorname{cn}(x) \operatorname{cn}(y)} \\ &= \frac{1 + k^2 \operatorname{sn}(x) \operatorname{cd}(x) \operatorname{sn}(y) \operatorname{cd}(y)}{1 - k^2 \operatorname{sn}(x) \operatorname{cd}(x) \operatorname{sn}(y) \operatorname{cd}(y)} = \frac{h(x-y)}{h(x+y)} \end{aligned}$$

Denoting $u(x,y) = \operatorname{sn}(x) \operatorname{cd}(x) \operatorname{sn}(y) \operatorname{cd}(y)$ and applying multiplicative telescoping, we deduce

$$\frac{1 + k^2 u(w-x, y-z)}{1 - k^2 u(w-x, y-z)} \frac{1 + k^2 u(w-y, z-x)}{1 - k^2 u(w-y, z-x)} \frac{1 + k^2 u(w-z, x-y)}{1 - k^2 u(w-z, x-y)} = 1$$

or explicitly

$$\begin{aligned} &\operatorname{sn}(w-x) \operatorname{cd}(w-x) \operatorname{sn}(y-z) \operatorname{cd}(y-z) \\ &+ \operatorname{sn}(w-y) \operatorname{cd}(w-y) \operatorname{sn}(z-x) \operatorname{cd}(z-x) \\ &+ \operatorname{sn}(w-z) \operatorname{cd}(w-z) \operatorname{sn}(x-y) \operatorname{cd}(x-y) \\ &+ k^4 \operatorname{sn}(w-x) \operatorname{cd}(w-x) \operatorname{sn}(y-z) \operatorname{cd}(y-z) \operatorname{sn}(w-y) \operatorname{cd}(w-y) \\ &\operatorname{sn}(z-x) \operatorname{cd}(z-x) \operatorname{sn}(w-z) \operatorname{cd}(w-z) \operatorname{sn}(x-y) \operatorname{cd}(x-y) = 0, \end{aligned}$$

an identity that can be qualified as "higher-order Jacobi identity."

[2] notice the typo in NIST entry [10, 22.5(ii)]

Example 4. Take the elementary function $h(t) = e^{-\frac{t^2}{4}}$ so that

$$f(x,y) = \frac{h(x-y)}{h(x+y)} = e^{-\frac{(x-y)^2-(x+y)^2}{4}} = e^{xy}$$

and multiplicative telescoping yields

$$e^{(w-x)(y-z)}e^{(w-y)(z-x)}e^{(w-z)(x-y)} = 1,$$

producing the identity $(w-x)(y-z) + (w-y)(z-x) + (w-z)(x-y) = 0$. Now performing the substitutions

$$x \to \cot x, y \to \cot y, z \to \cot z, w \to \cot w \tag{10}$$

and utilizing the addition formula

$$\cot a - \cot b = \frac{\sin(a-b)}{\sin a \sin b},$$

we recover Ptolemy's identity producing yet an another elementary proof of it.

6.3 Intermezzo: A problem in Copson's textbook.

The previous examples show how broadly this telescoping principle applies; Copson's elliptic problem supplies a classical instance closely tied to Jacobi's identity. Problem 43 in Copson's classic book [1] states:

`Prove that`

$$\frac{(1-k\operatorname{sn}u_1\operatorname{sn}u_2)(1-k\operatorname{sn}u_3\operatorname{sn}u_4)}{(1+k\operatorname{sn}u_1\operatorname{sn}u_2)(1+k\operatorname{sn}u_3\operatorname{sn}u_4)}$$

`is invariant under the transformation`

$$u_i' = \frac{u_1+u_2+u_3+u_4}{2} - u_i,\ 1 \le i \le 4. \tag{11}$$

`Deduce that`

$$\frac{1-k\operatorname{sn}(w-x)\operatorname{sn}(y-z)}{1+k\operatorname{sn}(w-x)\operatorname{sn}(y-z)}\frac{1-k\operatorname{sn}(w-y)\operatorname{sn}(z-x)}{1+k\operatorname{sn}(w-y)\operatorname{sn}(z-x)}\frac{1-k\operatorname{sn}(w-z)\operatorname{sn}(x-y)}{1+k\operatorname{sn}(w-z)\operatorname{sn}(x-y)} = 1. \tag{12}$$

In this section, we'll follow Copson's notation and write $\operatorname{sn}(u)$ for $\operatorname{sn}(u,k)$, unless the specification of the elliptic modulus k is necessary. Let us address the invariance property first: by Landen's ascending transformation [NIST 22.7.8],

$$\operatorname{dn}\left((1+k)u, \frac{2\sqrt{k}}{1+k}\right) = \frac{1-k\operatorname{sn}^2(u,k)}{1+k\operatorname{sn}^2(u,k)}$$

and using the addition theorem [NIST 22.8.3] for the elliptic sine function

$$\operatorname{sn}(u+v,k)\operatorname{sn}(u-v,k)=\frac{\operatorname{sn}^2(u,k)-\operatorname{sn}^2(v,k)}{1-k^2\operatorname{sn}^2(u,k)\operatorname{sn}^2(v,k)},$$

we deduce

$$1-k\operatorname{sn}u_1\operatorname{sn}u_2=\frac{\left(1-k\operatorname{sn}^2\left(\frac{u_1+u_2}{2}\right)\right)\left(1+k\operatorname{sn}^2\left(\frac{u_1-u_2}{2}\right)\right)}{1-k^2\operatorname{sn}^2(u,k)\operatorname{sn}^2(v,k)}$$

and, with $\tilde{k}=\frac{2\sqrt{k}}{1+k}$,

$$\frac{1-k\operatorname{sn}u_1\operatorname{sn}u_2}{1+k\operatorname{sn}u_1\operatorname{sn}u_2}=\frac{\left(1-k\operatorname{sn}^2\left(\frac{u_1+u_2}{2}\right)\right)\left(1+k\operatorname{sn}^2\left(\frac{u_1-u_2}{2}\right)\right)}{\left(1+k\operatorname{sn}^2\left(\frac{u_1+u_2}{2}\right)\right)\left(1-k\operatorname{sn}^2\left(\frac{u_1-u_2}{2}\right)\right)}=\frac{\operatorname{dn}\left((1+k)\frac{u_1+u_2}{2},\tilde{k}\right)}{\operatorname{dn}\left((1+k)\frac{u_1-u_2}{2},\tilde{k}\right)}$$

so that the expression

$$\frac{(1-k\operatorname{sn}u_1\operatorname{sn}u_2)(1-k\operatorname{sn}u_3\operatorname{sn}u_4)}{(1+k\operatorname{sn}u_1\operatorname{sn}u_2)(1+k\operatorname{sn}u_3\operatorname{sn}u_4)}=\frac{\operatorname{dn}\left((1+k)\frac{u_1+u_2}{2},\tilde{k}\right)}{\operatorname{dn}\left((1+k)\frac{u_1-u_2}{2},\tilde{k}\right)}\frac{\operatorname{dn}\left((1+k)\frac{u_3+u_4}{2},\tilde{k}\right)}{\operatorname{dn}\left((1+k)\frac{u_3-u_4}{2},\tilde{k}\right)}$$

is a function of the sums and differences $u_1\pm u_2$ and $u_3\pm u_4$ only. The identities

$$\begin{aligned}&u_1+u_2=u_3'+u_4',\ u_3+u_4=u_1'+u_2' \qquad \text{and}\\ &u_1-u_2=u_2'-u_1',\ u_3-u_4=u_4'-u_3',\end{aligned}$$

as a consequence of (11), produce the desired invariance. After the change of variables

$$w-x=u_1,\ y-z=u_2,\ldots$$

and expansion of (12) by clearing the denominators, we obtain *Jacobi's identity*: for all values of the elliptic modulus k,

$$\begin{aligned}&\operatorname{sn}(w-x)\operatorname{sn}(y-z)+\operatorname{sn}(w-y)\operatorname{sn}(z-x)+\operatorname{sn}(w-z)\operatorname{sn}(x-y)\\ &+k^2\operatorname{sn}(w-x)\operatorname{sn}(y-z)\operatorname{sn}(w-y)\operatorname{sn}(z-x)\operatorname{sn}(w-z)\operatorname{sn}(x-y)=0.\end{aligned}$$

In the article [7, Identity 96], this identity is attributed to Jacobi, with the comment [3] `"M. Jacobi found the following magnificent identity for elliptical functions".` Two specializations of this identity are of interest to us:

- in the case $k=1$, the Jacobi elliptic function $\operatorname{sn}(u,k)$ reduces to the hyperbolic tangent function and we recover the tangent version (7) of Gosper's identity
- in the case $k=0$, the Jacobi elliptic function $\operatorname{sn}(u,k)$ reduces to the trigonometric sine function, which results in Ptolemy's identity.

[3] "M. Jacobi a trouvé la magnifique identité suivante pour les fonctions elliptiques"

7 Fourier analysis: an additive three-pair identity.

The multiplicative formulation proves Jacobi's identity; Fourier analysis offers a complementary additive formulation. Recall Ptolemy's identity

$$\sin(w-x)\sin(y-z)+\sin(w-y)\sin(z-x)+\sin(w-z)\sin(x-y)=0.$$

Replacing each variable by an odd integer multiple of itself gives the corresponding identity for every odd Fourier frequency. This observation leads to the following result.

Proposition. *Suppose that*

$$f(x,y)=\sum_{n\geq 0} a_n \sin((2n+1)\pi x)\sin((2n+1)\pi y), \tag{13}$$

where the series converges pointwise at the arguments under consideration (for instance, it is sufficient that $\sum_{n\geq 0}|a_n|<\infty$*). Then*

$$f(w-x,y-z)+f(w-y,z-x)+f(w-z,x-y)=0. \tag{14}$$

Proof. For $n\geq 0$, Ptolemy's identity, applied to $(2n+1)\pi w,\dots,(2n+1)\pi z$, gives

$$\begin{aligned}&\sin((2n+1)\pi(w-x))\sin((2n+1)\pi(y-z))\\ &+\sin((2n+1)\pi(w-y))\sin((2n+1)\pi(z-x))\\ &+\sin((2n+1)\pi(w-z))\sin((2n+1)\pi(x-y))=0.\end{aligned}$$

Multiplying by a_n and summing over n proves (14). The convergence hypothesis justifies the summation. □

Example 5 (Jacobi's identity)**.** We recover Jacobi's identity (8) from (14). Let

$$f(x,y)=\log\frac{1-k\,\mathrm{sn}(x)\,\mathrm{sn}(y)}{1+k\,\mathrm{sn}(x)\,\mathrm{sn}(y)}.$$

We work in a domain where the logarithm is well-defined and the denominators do not vanish. Differentiating gives

$$\frac{\partial f}{\partial x}=-\frac{2k\,\mathrm{sn}(y)\,\mathrm{cn}(x)\,\mathrm{dn}(x)}{1-k^2\,\mathrm{sn}^2(x)\,\mathrm{sn}^2(y)}.$$

Using the addition formula

$$\mathrm{sn}(u+v)=\frac{\mathrm{sn}(u)\,\mathrm{cn}(v)\,\mathrm{dn}(v)+\mathrm{cn}(u)\,\mathrm{dn}(u)\,\mathrm{sn}(v)}{1-k^2\,\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)},$$

we obtain

$$\frac{\partial f}{\partial x}=-k\big(\mathrm{sn}(x+y)-\mathrm{sn}(x-y)\big).$$

Write $K = K(k)$ for the complete elliptic integral and let q denote the elliptic nome. The standard Fourier expansion for Jacobi elliptic function sn is

$$\operatorname{sn}(z) = \frac{2\pi}{kK} \sum_{n\geq 0} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}} \sin\Big((2n+1)\frac{\pi z}{2K}\Big).$$

For convenience, put $\varphi_n(z) = \sin\big((2n+1)\frac{\pi z}{2K}\big)$. It follows that

$$\begin{aligned}\frac{\partial f}{\partial x} &= -\frac{2\pi}{K} \sum_{n\geq 0} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}} \big(\varphi_n(x+y) - \varphi_n(x-y)\big) \\ &= -\frac{4\pi}{K} \sum_{n\geq 0} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}} \varphi_n(y) \cos\Big((2n+1)\frac{\pi x}{2K}\Big).\end{aligned}$$

Integrating with respect to x, and using $f(0,y) = 0$, yields

$$f(x,y) = -8 \sum_{n\geq 0} \frac{q^{n+\frac{1}{2}}}{(2n+1)(1-q^{2n+1})} \varphi_n(x)\varphi_n(y).$$

Let $X = x/(2K)$, $Y = y/(2K)$, and define $g(X,Y) = f(2KX, 2KY)$. Then g has the form (13), with

$$a_n = -\frac{8}{2n+1} \frac{q^{n+\frac{1}{2}}}{1-q^{2n+1}}.$$

Consequently, $g(W-X, Y-Z) + g(W-Y, Z-X) + g(W-Z, X-Y) = 0$. Returning to the original variables gives

$$f(w-x, y-z) + f(w-y, z-x) + f(w-z, x-y) = 0.$$

Exponentiating this logarithmic identity yields the product form of Jacobi's identity (8), wherever the logarithms are defined. The resulting product identity extends to other admissible arguments by analytic continuation.

Example 6 (Gosper's identity)**.** Consider

$$f(x,y) = \operatorname{arctanh}\big(\tan(\pi x)\tan(\pi y)\big).$$

This function is 1-periodic in each variable wherever it is defined. Therefore, it does not belong to the class in (13), whose elements are antiperiodic under a unit translation in either variable. Rather, its Fourier expansion reads

$$\operatorname{arctanh}\big(\tan(\pi x)\tan(\pi y)\big) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin(2\pi k x)\sin(2\pi k y).$$

Using Ptolemy's identity, we deduce

$$\begin{aligned}\operatorname{arctanh}(\tan(w-x)\tan(y-z)) &+ \operatorname{arctanh}(\tan(w-y)\tan(x-z)) \\ &+ \operatorname{arctanh}(\tan(w-z)\tan(x-y)) = 0.\end{aligned}$$

Where the relevant branches are defined, one may use the addition formula

$$\tanh(a+b+c) = \frac{\tanh a + \tanh b + \tanh c + \tanh a \tanh b \tanh c}{1 + \tanh a \tanh b + \tanh a \tanh c + \tanh b \tanh c}$$

to convert this identity into a rational identity for their hyperbolic tangents. This is the final step in deriving the tangent form (7) of Gosper's identity.

8 Theta functions: two trigonometric limits.

The elliptic identity of the preceding section has a useful theta-function form. In that form, it belongs to a family of genus-one addition identities associated with Fay's trisecant theory [9]. We will need no general trisecant theorem: Jacobi's elliptic identity gives the first theta identity directly, and a modular transformation gives a companion identity. Their limits as the nome tends to zero lead to two different trigonometric formulas, Ptolemy's identity and Gosper's identity.

We use the convention $q = e^{i\pi\tau}$, where $\operatorname{Im}\tau > 0$, and

$$\begin{aligned}
\theta_1(u,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin((2n+1)u),\\
\theta_2(u,q) &= 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos((2n+1)u),\\
\theta_3(u,q) &= \sum_{n\in\mathbb{Z}} q^{n^2}e^{2inu},\\
\theta_4(u,q) &= \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}e^{2inu}.
\end{aligned}$$

Thus the theta arguments below are ordinary differences of w, x, y, z; they have not silently been rescaled elliptic arguments. The three pairings of four variables will be $(\alpha_1,\beta_1) = (w-x, y-z), (\alpha_2,\beta_2) = (w-y, z-x), (\alpha_3,\beta_3) = (w-z, x-y)$. Let

$$U_j = \theta_1(\alpha_j,q)\theta_1(\beta_j,q), \qquad V_j^{(r)} = \theta_r(\alpha_j,q)\theta_r(\beta_j,q) \quad (r=2,4).$$

Notice that the order of each difference matters for U_j, because θ_1 is odd; it does not matter for $V_j^{(r)}$.

Proposition. *For $r = 4$ and for $r = 2$,*

$$U_1V_2^{(r)}V_3^{(r)} + V_1^{(r)}U_2V_3^{(r)} + V_1^{(r)}V_2^{(r)}U_3 + U_1U_2U_3 = 0. \tag{15}$$

In the limit $q \to 0$, the case $r = 2$ gives Gosper's identity while the case $r = 4$ gives Ptolemy's sine identity.

Proof. First suppose $0 < k < 1$, let $k' = \sqrt{1-k^2}$, and write $K = K(k)$ and $K' = K(k')$ for the complete elliptic integrals. Take $q = e^{-\pi K'/K}$ and put $z = 2Ku/\pi$. In our convention, Jacobi's theta representation reads [4, p. 492]

$$\operatorname{sn}(z,k) = \frac{\theta_3(0,q)}{\theta_2(0,q)}\frac{\theta_1(u,q)}{\theta_4(u,q)}, \qquad k^2 = \frac{\theta_2(0,q)^4}{\theta_3(0,q)^4}.$$

Apply the Jacobi four-point identity from the preceding section to the elliptic arguments $2K\alpha_j/\pi$, $2K\beta_j/\pi$:

$$\sum_{j=1}^{3} \operatorname{sn}\!\left(\frac{2K\alpha_j}{\pi},k\right)\operatorname{sn}\!\left(\frac{2K\beta_j}{\pi},k\right) + k^2\prod_{j=1}^{3}\operatorname{sn}\!\left(\frac{2K\alpha_j}{\pi},k\right)\operatorname{sn}\!\left(\frac{2K\beta_j}{\pi},k\right) = 0.$$

Divide by $[\theta_3(0,q)/\theta_2(0,q)]^2$. The coefficient of the six-factor quotient then becomes $k^2[\theta_3(0,q)/\theta_2(0,q)]^4 = 1$. Multiplying by $\prod_{j=1}^{3} V_j^{(4)}$ gives (15) for $r = 4$. This computation is made first where the denominators do not vanish; the resulting identity is an identity of holomorphic functions and holds at their zeros as well.

For completeness, we notice that the $r = 2$ case follows from the $r = 4$ case without a second elliptic-function substitution. Using the shorthand $\theta_j(u \mid \tau) = \theta_j(u, e^{i\pi\tau})$ and put $\tau' = -1/\tau$. The modular transformations in this convention are

$$\theta_1(u \mid \tau) = i(-i\tau)^{-1/2}e^{-iu^2/(\pi\tau)}\theta_1(u/\tau \mid \tau'),$$
$$\theta_4(u \mid \tau) = (-i\tau)^{-1/2}e^{-iu^2/(\pi\tau)}\theta_2(u/\tau \mid \tau').$$

Each term of the $r = 4$ identity contains all six arguments once, so its Gaussian multiplier is the same. Its first three terms contain two θ_1 factors each, and its final term contains six. Therefore , their remaining multipliers are $i^2 = i^6 = -1$. After canceling the common nonzero factor, we obtain the identity $r = 2$ at τ' with all arguments divided by τ. Since the variables and τ' are arbitrary, this proves the second case. Holomorphic continuation gives both identities for every $|q| < 1$. □

The difference between the two identities becomes visible when $q \to 0$ through positive values. For fixed u, we have

$$\theta_1(u,q) = 2q^{1/4}\sin u + O(q^{9/4}), \qquad \theta_2(u,q) = 2q^{1/4}\cos u + O(q^{9/4}),$$
$$\theta_4(u,q) = 1 + O(q).$$

Write $S_j = \sin\alpha_j \sin\beta_j$ and $C_j = \cos\alpha_j\cos\beta_j$. The left-hand side L_4 of (15) for $r = 4$ satisfies

$$L_4 = 4q^{1/2}(S_1 + S_2 + S_3) + O(q^{3/2}).$$

Consequently its first nonzero-order relation is Ptolemy's sine identity,

$$\sin(w-x)\sin(y-z) + \sin(w-y)\sin(z-x) + \sin(w-z)\sin(x-y) = 0.$$

For $r = 2$, every term has six factors of order $q^{1/4}$. The corresponding left-hand side satisfies

$$L_2 = 64q^{3/2}\big(S_1C_2C_3 + C_1S_2C_3 + C_1C_2S_3 + S_1S_2S_3\big) + O(q^{7/2}).$$

This coefficient is Gosper's identity, with an overall minus sign if Gosper's expression is written in the orientation used earlier in this article. Indeed, put

$$\begin{aligned} A &= \sin(x-w)\sin(y-z), \qquad & a &= \cos(x-y)\cos(z-w),\\ B &= \sin(x-y)\sin(z-w), \qquad & b &= \cos(x-z)\cos(y-w),\\ C &= \sin(y-w)\sin(z-x), \qquad & c &= \cos(x-w)\cos(y-z). \end{aligned}$$

Then $(S_1, S_2, S_3) = (-A, -C, -B)$ and $(C_1, C_2, C_3) = (c, b, a)$. Hence

$$\begin{aligned} L_4 &= -4q^{1/2}(A + B + C) + O(q^{3/2}),\\ L_2 &= -64q^{3/2}(ABC + Aab + Bbc + Cca) + O(q^{7/2}). \end{aligned}$$

Dividing the two theta identities by their indicated leading powers of q recovers, respectively, Ptolemy's identity and Gosper's identity. The same theta-function framework thus connects them, while retaining the distinction between their trigonometric limits.

9 Rank-two matrices and Ptolemy-type identities.

The six-variable relation is another manifestation of Ptolemy's identity, and its rank-two proof suggests a Pfaffian extension to any even number of variables. Also, the rank-two Pfaffians extend Ptolemy's identity; Schur's Pfaffian identity provides the corresponding route to a Gosper-type extension.

9.1 The $6-$variables case.

For arbitrary complex parameters u, v, w, x, y, z, we refer to this version of Ptolemy's identity

$$\begin{aligned} &\sin(u-x)\sin(v-y)\sin(w-z) + \sin(x-u)\sin(v-z)\sin(w-y)\\ &+\sin(y-u)\sin(v-x)\sin(w-z) + \sin(u-y)\sin(v-z)\sin(w-x)\\ &+\sin(u-z)\sin(v-x)\sin(w-y) + \sin(z-u)\sin(v-y)\sin(w-x) = 0. \end{aligned} \tag{16}$$

We first establish its polynomial analog. Let P denote the expression obtained by replacing each sine of a difference in (16) with that difference:

$$\begin{aligned} P = &(u-x)(v-y)(w-z) + (x-u)(v-z)(w-y)\\ &+ (y-u)(v-x)(w-z) + (u-y)(v-z)(w-x)\\ &+ (u-z)(v-x)(w-y) + (z-u)(v-y)(w-x). \end{aligned} \tag{17}$$

Pairing consecutive summands and using $x-u=-(u-x)$, $y-u=-(u-y)$, and $z-u=-(u-z)$ gives

$$P=(v-w)\big[(u-x)(y-z)+(u-y)(z-x)+(u-z)(x-y)\big]=0. \qquad (18)$$

There is also a residue proof of (18). Assume first that x,y,z are pairwise distinct, and consider u,v,w,x,y,z as parameters. For the independent complex variable t, let

$$R(t)=(v-w)(x-y)(y-z)(z-x)\frac{t-u}{(t-x)(t-y)(t-z)}.$$

The only possible finite poles are simple poles at x,y,z (some may be removable). Direct cancellation yields

$$\begin{aligned}\mathrm{Res}_{t=x}\,R(t)&=(v-w)(u-x)(y-z),\\ \mathrm{Res}_{t=y}\,R(t)&=(v-w)(u-y)(z-x),\\ \mathrm{Res}_{t=z}\,R(t)&=(v-w)(u-z)(x-y).\end{aligned}$$

Since $R(t)=O(t^{-2})$ as $t\to\infty$, its residue at infinity is zero. The residue theorem therefore sets the sum of these three residues equal to zero, exactly as in (18). Finally, P is a polynomial in all six parameters; because it vanishes whenever x,y,z are pairwise distinct, it also vanishes when some of them coincide. Thus (18) holds without distinctness assumptions.

Apply the polynomial identity simultaneously to the six values

$$E_a=e^{2\imath a},\qquad a\in\{u,v,w,x,y,z\}.$$

Every difference becomes $E_a-E_b=2\imath\,e^{\imath(a+b)}\sin(a-b)$. Each of the six products in (17) contains every parameter exactly once. Hence every product acquires the same nonzero exponential multiplier

$$(2\imath)^3e^{\imath(u+v+w+x+y+z)}.$$

Dividing by it gives (16). Equivalently, its paired form is

$$\begin{aligned}\sin(v-w)\sin(u-x)\sin(y-z)+\sin(v-w)\sin(u-y)\sin(z-x)\\ +\sin(v-w)\sin(u-z)\sin(x-y)=0. \qquad (19)\end{aligned}$$

The bracket in (19) is the signed four-point chord (Plücker) relation. For real parameters $u<x<y<z<u+\pi$, the points $e^{2\imath u},e^{2\imath x},e^{2\imath y},e^{2\imath z}$ occur in cyclic order, and their chord lengths are $2|\sin(a-b)|$. In this order, the middle term has the opposite sign from the other two, so the bracket is precisely Ptolemy's equality: the product of the diagonals equals the sum of the products of opposite sides. The signed identity itself requires no ordering or reality assumptions.

We can propose a more algebraic proof of (16) that will be useful later: with notations $u=a_1,v=a_2,w=a_3,x=b_1,y=b_2$ and $z=b_3$, the left-hand side coincides with the determinant of the (3×3) matrix M with entries

$$M_{i,j}=\sin\left(a_i-b_j\right),\ 1\le i,j\le3.$$

As a consequence of $\sin(a+b) = \sin a \cos b + \cos a \sin b$, matrix M factors as

$$M = \begin{bmatrix} \sin a_1 & -\cos a_1 \\ \sin a_2 & -\cos a_2 \\ \sin a_3 & -\cos a_3 \end{bmatrix} \begin{bmatrix} \cos b_1 & \cos b_2 & \cos b_3 \\ \sin b_1 & \sin b_2 & \sin b_3 \end{bmatrix}.$$

Since each factor is a matrix with rank less than 2, the (3×3) matrix M itself has rank ≤ 2 so that its determinant is zero. This condition is equivalent to (16).

9.2 Algebraic proofs in the $2m$-variate case.

At this point, we wonder if there exist versions of Ptolemy and Gosper's identity with an arbitrary even number of variables. Let us consider Ptolemy's identity first.

Proposition. *Let $\{x_1, \ldots, x_{2m}\}$ a set of $2m$ variables with $m \geq 2$ and $\mathcal{M}_{2m}$ the set of perfect matchings M of $\{1, 2, \ldots, 2m\}$, then*

$$\sum_{M \in \mathcal{M}_{2m}} \varepsilon(M) \prod_{\{i,j\} \in M} (x_i - x_j) = 0 \tag{20}$$

and

$$\sum_{M \in \mathcal{M}_{2m}} \varepsilon(M) \prod_{\{i,j\} \in M} \sin(x_i - x_j) = 0. \tag{21}$$

Proof. Identity (20) is well-known: the left-hand side is the Pfaffian of the $(2m \times 2m)$ skew-symmetric matrix N with entry $N_{i,j} = x_i - x_j$. Since matrix N can be expressed as the difference of two rank-one operators (with $\mathbf{1}^T = [1, 1, \ldots 1]$)

$$N = \mathbf{x}\mathbf{1}^T - \mathbf{1}\mathbf{x}^T,$$

the rank of N is at most 2 and the Pfaffian of N vanishes as soon as $m \geq 2$. Identity (21) can be deduced by substituting each x_i with $\tan x_i$ in N and using the addition formula

$$\tan x_i - \tan x_j = \frac{\sin(x_i - x_j)}{\cos x_i \cos x_j}.$$

An alternate proof of identity (21) that bypasses its linear version (20) considers directly the $(2m \times 2m)$ matrix S with entries

$$S_{i,j} = \sin(x_i - x_j), \ 1 \leq i, j \leq 2m.$$

Since $\sin(x_i - x_j) = s_i c_j - c_i s_j$, we deduce the factorization of S as the difference of two rank-one operators $S = \mathbf{s}\mathbf{c}^t - \mathbf{c}\mathbf{s}^t$, so that the rank of S is at most 2 and its Pfaffian vanishes as soon as $m \geq 2$. □

10 Schur's Pfaffian identity and the Gosper-type extension.

In the case of Gosper's identity, we propose a proof that relies on an identity that is well-known in combinatorics. Let's revive Schur's Pfaffian identity which says that for variables $u_1, \ldots, u_{2m}$, we have

$$\mathrm{Pf}\left[\frac{u_j - u_k}{u_j + u_k}\right]_{1\le j<k\le 2m} = \prod_{1\le j<k\le 2m} \frac{u_j - u_k}{u_j + u_k}$$

Set $u_k = e^{2\imath\theta_k}$ for some real numbers $\theta_1, \ldots, \theta_{2m}$. Then

$$\frac{u_j - u_k}{u_j + u_k} = \imath \tan(\theta_j - \theta_k).$$

Since $\binom{2m}{2} = m(2m-1)$, the powers of i on both sides cancel, giving

$$\mathrm{Pf}[\tan(\theta_j - \theta_k)] = \prod_{j<k} \tan(\theta_j - \theta_k).$$

Define $s_{jk} = \sin(\theta_j - \theta_k)$, $c_{jk} = \cos(\theta_j - \theta_k)$ and $t_{jk} = \tan(\theta_j - \theta_k)$. Then, whenever all $c_{jk} \neq 0$,

$$\mathrm{Pf}[t_{jk}]_{1\le j<k\le 2m} = \prod_{1\le j<k\le 2m} t_{jk}.$$

Equivalently, after clearing denominators, one obtains the sine-cosine identity

$$\sum_M \varepsilon(M) \left(\prod_{\{j,k\}\in M} s_{jk}\right)\left(\prod_{\{j,k\}\notin M} c_{jk}\right) - \prod_{1\le j<k\le 2m} s_{jk} = 0$$

where the sum is over all perfect matchings M of $\{1, 2, \ldots, 2m\}$, and $\varepsilon(M)$ is the Pfaffian sign rules.

Example For $2m = 4$, the perfect matchings are $\{12, 34\}, \{13, 24\}, \{14, 23\}$, with Pfaffian signs $+$, $-$, $+$. Thus the identity becomes

$$s_{12}s_{34}c_{13}c_{14}c_{23}c_{24} - s_{13}s_{24}c_{12}c_{14}c_{23}c_{34} + s_{14}s_{23}c_{12}c_{13}c_{24}c_{34} - s_{12}s_{13}s_{14}s_{23}s_{24}s_{34} = 0.$$

Taking $\theta_1 = x, \theta_2 = y, \theta_3 = z, \theta_4 = w$ recovers Gosper's original six-factor identity after rewriting the sine factors such as $\sin(z - x) = -\sin(x - z)$.

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References

[1] E.T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, 1935

[2] W. Gosper, Strip Mining in the Abandoned Orefields of Nineteenth Century Mathematics, in: D. V. Chudnovsky and R. D. Jenks (eds.), Computers in Mathematics, Lecture Notes in Pure and Applied Mathematics, vol. 125, 1990, p. 261

[3] F.J. Dyson, Statistical theory of the energy levels of complex systems. I, Journal of Mathematical Physics 3 (1962) 140–156

[4] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, 4th Edition, Cambridge University Press, 1927

[5] S. Cooper, P. C. Toh, Determinant identities for theta functions, J. Math. Anal. Appl. 347 (2008) 1–7

[6] L.-C. Shen, On the Additive Formulae of the Theta Functions and a Collection of Lambert Series Pertaining to the Modular Equations of Degree 5, Trans. Amer. Math. Soc. 345 (1994), 323–345

[7] Recueil de formules relatives aux fonctions circulaires et logarithmiques. Suite. Nouvelles Annales de Mathématiques, 1re série, tome 7, (1848), 269-271

[8] Material from Bill Gosper's Computers and Math Talk, M.I.T., 1989, https://oeis.org/A309204/a309204.pdf

[9] J.D. Fay, Theta Functions on Riemann Surfaces, Lecture Notes in Mathematics 352, Springer, 1973

[10] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.

[11] J. Aczél, Lectures on Functional Equations and Their Applications, Mathematics in Science and Engineering, Vol. 19, Academic Press, New York, London, 1966