CONGRUENCES FOR SUMS OF MACMAHON'S q-CATALAN POLYNOMIALS

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ABSTRACT. One variant of the q-Catalan polynomials is defined in terms of Gaussian polynomials by $\mathcal{C}_k(q) = \begin{bmatrix}2k\\k\end{bmatrix}_q - q\begin{bmatrix}2k\\k+1\end{bmatrix}_q$. Liu studied congruences of the form $\sum_{k=0}^{n-1} q^k \mathcal{C}_k$ modulo the cyclotomic polynomial $\Phi_n(q)^2$, provided that $n \equiv \pm 1 \pmod 3$. Apparently the case $n \equiv 0 \pmod 3$ has been missing from the literature. It is our primary purpose to fill this gap by the current work. In addition, we discuss certain fascinating link to Dirichlet character sum identities.

1. INTRODUCTION

There are several possible q-analogs of the Catalan numbers $\frac{1}{k+1}\binom{2k}{k}$. Here we consider the MacMahon's q-Catalan polynomials which are defined in terms of the q-binomial coefficients (Gaussian polynomials) as

$$\mathcal{C}_k(q) := \frac{1-q}{1-q^{k+1}}\begin{bmatrix}2k\\k\end{bmatrix}_q = \begin{bmatrix}2k\\k\end{bmatrix}_q - q\begin{bmatrix}2k\\k+1\end{bmatrix}_q.$$

The first few of these q-Catalan are $\mathcal{C}_0(q) = \mathcal{C}_1(q) = 1$, $\mathcal{C}_2(q) = 1 + q^2$ and $\mathcal{C}_3(q) = 1 + q^2 + q^3 + q^4 + q^6$. Notice that $\mathcal{C}_k(q)$ is a polynomial in q and it reduces to the ordinary Catalan number as $q \to 1$. Moreover, $\mathcal{C}_k(q)$ has a natural enumerative meaning. Indeed MacMahon [4, Vol. 2, p. 214] established that

$$\mathcal{C}_k(q) = \sum_w q^{\mathrm{maj}(w)}$$

where w ranges over all ballot sequences $a_1a_2\cdots a_{2k}$ (i. e. any permutation of the multiset $\{0^k, 1^k\}$ such that in any subword $a_1a_2\cdots a_i$, there are at least as many 0s as there are 1s) and

$$\mathrm{maj}(w) := \sum_{\{i: a_i > a_{i+1}\}} i$$

is the *major index* of w (see also the survey [1, Section 3] and [7, Problem A43]). In the present work, however, we focus on a problem of number-theoretic interest: congruences. The second author in [8, Theorem 6.1] achieved the following result

$$\sum_{k=0}^{n-1} q^k \mathcal{C}_k(q) \equiv \begin{cases} q^{\lfloor \frac{n}{3} \rfloor} & \text{if } n \equiv 0, 1 \pmod 3 \\ -1 - q^{\frac{2n-1}{3}} & \text{if } n \equiv 2 \pmod 3 \end{cases} \pmod{\Phi_n(q)}$$

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where

$$\Phi_n(q)=\prod_{\substack{1\le k\le n\\(n,k)=1}}(q-e^{\frac{2k\pi i}{n}})$$

denotes the n^{th}-cyclotomic polynomial.

Afterwards, a stronger version modulo $\Phi_n(q)^2$, has been proved by J.-C. Liu in [2, Theorem 1],

$$\sum_{k=0}^{n-1}q^k\mathcal{C}_k(q)\equiv\begin{cases}q^{\frac{n^2-1}{3}}-\frac{n-1}{3}(q^n-1) & \text{if } n\equiv 1 \pmod 3,\\ -q^{\frac{n^2-1}{3}}-q^{\frac{n(2n-1)}{3}} & \text{if } n\equiv 2 \pmod 3,\end{cases}\pmod{\Phi_n(q)^2}$$

where the case $n\equiv 0 \pmod 3$ is *not* covered. Our main aim at present is to fill this gap as stated next.

Theorem 1.1. *If n is a positive integer divisible by 3, then*

$$\sum_{k=0}^{n-1}q^k\mathcal{C}_k(q)\equiv q^{\frac{n(2n+1)}{3}}+\frac{1}{3}(q^n-1)\Big(2+(n+1)q^{\frac{2n}{3}}\Big)\pmod{\Phi_n(q)^2}.$$

As we will explain in more detail below, the above theorem holds as soon as we prove the following more manageable identity, which is of interest in its own right.

Theorem 1.2. *If n is a positive integer divisible by 3 and q is a primitive n^{th}-root of unity, then*

$$\sum_{k=1}^{\frac{n}{3}}\frac{(-1)^kq^{\frac{k(3k-1)}{2}}}{1-q^{3k-1}}+\sum_{k=1}^{\frac{n}{3}-1}\frac{(-1)^kq^{\frac{k(3k+5)}{2}}}{1-q^{3k}}=\frac{1}{6}\left(2+(n+1)q^{\frac{2n}{3}}\right).\tag{1}$$

Notice that, according to [2, Lemma 3], our Theorem 1.2 mirrors that of

$$\sum_{k=1}^{\lfloor\frac{n}{3}\rfloor}\frac{(-1)^kq^{\frac{k(3k-1)}{2}}}{1-q^{3k-1}}+\sum_{k=1}^{\lfloor\frac{n-1}{3}\rfloor}\frac{(-1)^kq^{\frac{k(3k+5)}{2}}}{1-q^{3k}}\equiv\begin{cases}-\frac{n-1}{6} & \text{if } n\equiv 1 \pmod 3,\\ 0 & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

The remainder of the paper is organized as follows. In Section 2, we present a reduction of our main result Theorem 1.1 into Theorem 1.2. Section 3 contains preliminary results which we need towards the proof of Theorem 1.2. Sections 4 and 5 split up Theorem 1.2 according to the parity of n and we provide the corresponding proofs respectively therein. Finally, in Section 6 we consider a conversion of one particular identity coming from (1), into a trigonometric format and a remarkable implication of it in the language of character sums.

2. Reducing Theorem 1.1 to Theorem 1.2

We recall that the Gaussian q-binomial coefficients are defined as

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{cases}\dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0\le k\le n,\\ 0 & \text{otherwise,}\end{cases}$$

where the q-shifted factorial is given by $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\ge 1$ and $(a;q)_0=1$.

By [3, Theorem 1.2],

$$\sum_{k=0}^{n-1} q^k \begin{bmatrix}2k\\k\end{bmatrix}_q \equiv \left(\frac{n}{3}\right) q^{\frac{n^2-1}{3}} \pmod{\Phi_n(q)^2} \tag{2}$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. In the same vain, we also revive the identity [9, Theorem 4.2],

$$\sum_{k=0}^{n-1} q^{k+1} \begin{bmatrix}2k\\k+1\end{bmatrix}_q = \sum_{k=1}^{n} \left(\frac{k-1}{3}\right) q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)} \begin{bmatrix}2n\\n+k\end{bmatrix}_q.$$

Let $1 \leq k \leq n-1$. Then, the q-analog [6, Theorem 2.2]

$$\begin{bmatrix}an+b\\cn+d\end{bmatrix}_q \equiv \binom{a}{c} \begin{bmatrix}b\\d\end{bmatrix}_q \pmod{\Phi_n(q)}$$

of Lucas' classical binomial congruence combined with $(1-q^n) \equiv 0 \pmod{\Phi_n(q)}$, and the fact that

$$\begin{bmatrix}n-1\\k-1\end{bmatrix}_q = \prod_{j=1}^{k-1} \frac{1-q^{n-j}}{1-q^j} = q^{-\frac{k(k-1)}{2}} \prod_{j=1}^{k-1} \frac{q^j-q^n}{1-q^j} \equiv (-1)^{k-1} q^{-\frac{k(k-1)}{2}} \pmod{\Phi_n(q)}$$

immediately imply that

$$\begin{aligned}\begin{bmatrix}2n\\n+k\end{bmatrix}_q &= \frac{1-q^{2n}}{1-q^{n+k}} \begin{bmatrix}2n-1\\n+k-1\end{bmatrix}_q \equiv (1-q^n) \cdot \frac{2}{1-q^k} \binom{1}{1} \begin{bmatrix}n-1\\k-1\end{bmatrix}_q \\ &\equiv (q^n-1) \cdot \frac{2(-1)^k q^{-\frac{k(k-1)}{2}}}{1-q^k} \pmod{\Phi_n(q)^2}.\end{aligned}$$

Therefore, we find reliable verity to declare that

$$\begin{aligned}\sum_{k=0}^{n-1} q^{k+1} \begin{bmatrix}2k\\k+1\end{bmatrix}_q &\equiv \sum_{k=0}^{n-1} \left(\frac{k-1}{3}\right) q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)} \begin{bmatrix}2n\\n+k\end{bmatrix}_q \\ &\equiv 2(q^n-1) \sum_{k=1}^{n-1} \left(\frac{k-1}{3}\right) q^{\frac{1}{3}\left(2k^2-k\left(\frac{k-1}{3}\right)\right)} \frac{(-1)^k q^{-\frac{k(k-1)}{2}}}{1-q^k} \\ &\equiv -2(q^n-1) \left(\sum_{k=1}^{\lfloor \frac{n}{3} \rfloor} \frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{1-q^{3k-1}} + \sum_{k=1}^{\lfloor \frac{n-1}{3} \rfloor} \frac{(-1)^k q^{\frac{k(3k+5)}{2}}}{1-q^{3k}} \right)\end{aligned}$$

holds modulo $\Phi_n(q)^2$.

Hence, by putting this congruence together with (2) into the very definition of $\mathcal{C}_k(q)$, we easily obtain that, when n is divisible by 3, Theorem 1.1 is indeed equivalent to Theorem 1.2.

3. Preparing our proof of Theorem 1.2

Henceforth, we replace n with $3n$ so that our target in (1) amounts to proving

$$\sum_{k=1}^{n} \frac{(-1)^k q^{\frac{k(3k-1)}{2}}}{1-q^{3k-1}} + \sum_{k=1}^{n-1} \frac{(-1)^k q^{\frac{k(3k+5)}{2}}}{1-q^{3k}} = \frac{1}{3} + \frac{3n+1}{6} q^{2n}. \tag{3}$$

In order to establish this identity, we need the next two results.

Lemma 3.1. *We have that for any complex number z,*

$$\sum_{k=1}^{n}\frac{(-1)^k z^{\frac{k(3k-1)}{2}}}{1-z^{3k-1}}+\sum_{k=1}^{n-1}\frac{(-1)^k z^{\frac{k(3k+5)}{2}}}{1-z^{3k}}$$
$$=\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\frac{z^{\frac{k(3n+2)}{2}}}{1+z^{\frac{3k}{2}}}+\frac{1}{2}\sum_{k=1}^{n-1}\frac{(-1)^k z^{\frac{k(3n+2)}{2}}}{1-z^{\frac{3k}{2}}}$$
$$+\sum_{k=1}^{n}\frac{1}{1-z^{3k-1}}-\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{1-z^{3k-2}}-\frac{2n-1+(-1)^n}{4}. \tag{4}$$

Proof. Employing partial fractions and after further rearrangement, we obtain

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^k z^{\frac{k(3k-1)}{2}}}{1-z^{3k-1}}&=\frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k z^{\frac{k(3k-1)}{2}}}{1-z^{\frac{3k-1}{2}}}+\frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k z^{\frac{k(3k-1)}{2}}}{1+z^{\frac{3k-1}{2}}}\\
&=\frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k((z^{\frac{3k-1}{2}})^k-1+1)}{1-z^{\frac{3k-1}{2}}}-\frac{1}{2}\sum_{k=1}^{n}\frac{-(-z^{\frac{3k-1}{2}})^k+1-1}{1-(-z^{\frac{3k-1}{2}})}\\
&=-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=0}^{k-1}((-1)^k+(-1)^j)z^{\frac{j(3k-1)}{2}}\\
&\qquad+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{1+z^{\frac{3k-1}{2}}}+\frac{(-1)^k}{1-z^{\frac{3k-1}{2}}}\right)\\
&=-\frac{1}{2}\sum_{k=1}^{n}\sum_{j=0}^{k-1}((-1)^k+(-1)^j)z^{\frac{j(3k-1)}{2}}\\
&\qquad+\sum_{k=1}^{n}\frac{1}{1-z^{3k-1}}-\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{1-z^{3k-2}}.\end{aligned}$$

Continuing with additional algebraic manipulation leads to

$$\sum_{k=1}^{n}\sum_{j=0}^{k-1}((-1)^k+(-1)^j)z^{\frac{j(3k-1)}{2}}=\sum_{j=0}^{n-1}\sum_{k=j+1}^{n}((-1)^k+(-1)^j)z^{\frac{j(3k-1)}{2}}$$
$$=\frac{2n-1+(-1)^n}{2}+2\sum_{j=1}^{n-1}\frac{(-1)^j z^{\frac{j(3j+5)}{2}}}{1-z^{3j}}+\sum_{j=1}^{n-1}\left(\frac{(-1)^n z^{\frac{j(3n+2)}{2}}}{1+z^{\frac{3j}{2}}}-\frac{(-1)^j z^{\frac{j(3n+2)}{2}}}{1-z^{\frac{3j}{2}}}\right).$$

Combining the last two calculations, we find (4). □

Lemma 3.2. *If α is a primitive m^{th}-root of unity, we have*

$$\sum_{k=1}^{m}\frac{1}{1-z^{-1}\alpha^k}=\frac{m}{1-z^{-m}}. \tag{5}$$

Proof. We introduce the function $f(z):=z^m-1=\prod_{k=1}^{m}(z-\alpha^k)$. Then, taking the logarithmic derivative, we obtain

$$\frac{f'(z)}{f(z)}=\sum_{k=1}^{m}\frac{1}{z-\alpha^k}$$

which means

$$\frac{m}{1-z^{-m}}=\sum_{k=1}^{m}\frac{1}{1-z^{-1}\alpha^k}.$$

□

We set $q=\exp(\frac{2\pi ij}{3n})$ with $\gcd(j,3n)=1$. If we apply (5) with $\alpha=q^3$ and $z=q$ and $m=n$, there holds

$$\sum_{k=1}^{n}\frac{1}{1-q^{3k-1}}=\frac{n}{1-q^{-n}}=\frac{n}{3}(1-q^n). \tag{6}$$

Combining (6) and by using the right-hand side of (4) with $z=q$, we put the central declaration (3) in a form that is more convenient for our method of proof:

$$\begin{aligned}&\frac{(-1)^{n-1}}{2}\sum_{k=1}^{n-1}\frac{q^{\frac{k(3n+2)}{2}}}{1+q^{\frac{3k}{2}}}+\frac{1}{2}\sum_{k=1}^{n-1}\frac{(-1)^k q^{\frac{k(3n+2)}{2}}}{1-q^{\frac{3k}{2}}}+\frac{n}{3}(1-q^n)\\ &-\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{1}{1-q^{3k-2}}-\frac{2n-1+(-1)^n}{4}=\frac{1}{3}+\frac{3n+1}{6}\,q^{2n}.\end{aligned} \tag{7}$$

Next, we proceed to study (7) by distinguishing two cases: $n=2N$ and $n=2N-1$. This allows us to circumvent fractional powers of q.

(a) If $n=2N$ then $q^{3N}=(-1)^j=-1$ because j is odd. Engaging with some algebraic simplifications, we find that (7) actually tantamount

$$\begin{aligned}&q^{2N}\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}+\sum_{k=1}^{N}\frac{q^{2k-1}}{1-q^{6k-3}}+\frac{2N}{3}(1-q^{2N})\\ &\qquad-\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}-N=\frac{1}{3}-\left(N+\frac{1}{6}\right)q^N.\end{aligned}$$

(b) If $n=2N-1$ then we determine that (7) is equivalent to

$$\begin{aligned}&q^{2N-1}\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}+\sum_{k=1}^{N-1}\frac{q^{2k}}{1-q^{6k}}+\frac{2N-1}{3}(1-q^{2N-1})\\ &\qquad-\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}-(N-1)=\frac{1}{3}+\left(N-\frac{1}{3}\right)q^{2(2N-1)}.\end{aligned}$$

In the next two sections, we intend to furnish the proofs for these two cases.

4. Proof of the case $n=2N$

The condition $\gcd(j,6N)=1$ forces $j=\pm1 \pmod 6$. We set $\omega:=q^N$ so that $1-\omega+\omega^2=0$ and $\omega^3=-1$. Therefore, it suffices to show the following.

Lemma 4.1. *We have that*

$$\omega^2\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}+\sum_{k=1}^{N}\frac{q^{2k-1}}{1-q^{6k-3}}-\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}=-\frac{N}{3}(1+\omega)+\frac{1}{3}-\frac{\omega}{6}. \tag{8}$$

Proof. We find it convenient to express our claim in terms of the quantities

$$A_1=\sum_{k=1}^{N-1}\frac{1}{1-q^k},\quad A_2=\sum_{k=1}^{N-1}\frac{1}{1-\omega q^k},\quad A_3=\sum_{k=1}^{N-1}\frac{1}{1-\omega^2 q^k},$$
$$A_4=\sum_{k=1}^{N-1}\frac{1}{1+q^k},\quad A_5=\sum_{k=1}^{N-1}\frac{1}{1+\omega q^k},\quad A_6=\sum_{k=1}^{N-1}\frac{1}{1+\omega^2 q^k}.$$

(i) By partial fraction decomposition

$$\frac{6x}{1-x^6}=\frac{1}{1-x}-\frac{\omega}{1-\omega^2 x}+\frac{\omega^2}{1+\omega x}-\frac{1}{1+x}+\frac{\omega}{1+\omega^2 x}-\frac{\omega^2}{1-\omega x}. \tag{9}$$

Hence, taking $x=q^k$ results in

$$6\sum_{k=1}^{N-1}\frac{q^k}{1-q^{6k}}=A_1-\omega^2A_2-\omega A_3-A_4+\omega^2A_5+\omega A_6.$$

(ii) Again by partial fraction decomposition

$$\frac{3x}{1-x^3}=\frac{1}{1-x}-\frac{\omega}{1-\omega^2 x}+\frac{\omega^2}{1+\omega x}. \tag{10}$$

Thus, the choice $x=q^{2k-1}$ gives

$$3\sum_{k=1}^{N}\frac{q^{2k-1}}{1-q^{6k-3}}=B_1-\omega B_2+\omega^2 B_3,$$

where

$$B_1=\sum_{k=1}^{N}\frac{1}{1-q^{2k-1}},\quad B_2=\sum_{k=1}^{N}\frac{1}{1-\omega^2 q^{2k-1}},\quad B_3=\sum_{k=1}^{N}\frac{1}{1+\omega q^{2k-1}}.$$

It is easy to check the properties $B_2=\frac{N}{2}$ and $B_1+B_3=N$ directly from

$$2\,\mathrm{Re}(B_2)=B_2+\overline{B_2}=\sum_{k=1}^{N}\frac{1}{1+\omega^{-1}q^{2k-1}}+\sum_{k=1}^{N}\frac{1}{1+\omega q^{1-2k}},$$
$$B_1+B_3=\sum_{k=1}^{N}\frac{1}{1-q^{2k-1}}+\sum_{k=1}^{N}\frac{1}{1+q^N q^{2N+2-2k-1}}.$$

Consequently, we obtain

$$3\sum_{k=1}^{N}\frac{q^{2k-1}}{1-q^{6k-3}}=B_1-\frac{\omega N}{2}+\omega^2(N-B_1)=(2-\omega)\left(B_1-\frac{N}{2}\right).$$

Moreover, we recognize that

$$B_1=\sum_{k=1}^{2N-1}\frac{1}{1-q^k}-\sum_{k=1}^{N-1}\frac{1}{1-q^{2k}}=A_1+\frac{1}{1-q^N}+A_2-\frac{A_1+A_4}{2}$$
$$=\frac{A_1}{2}+A_2-\frac{A_4}{2}+\omega.$$

(iii) Introducing the values

$$C_1 := \sum_{k=1}^{N} \frac{1}{1-q^{3k-1}}, \quad C_2 := \sum_{k=1}^{N} \frac{1}{1-q^{3k-2}},$$

we gather the property that (where we use a partial fraction of $\frac{1}{1-x^3}$)

$$\begin{aligned} C_1 + C_2 &= \sum_{k=1}^{3N-1} \frac{1}{1-q^k} - \sum_{k=1}^{N-1} \frac{1}{1-q^{3k}} \\ &= A_1 + \frac{1}{1-q^N} + A_2 + \frac{1}{1-q^{2N}} + A_3 - \frac{A_1 + A_3 + A_5}{3} \\ &= \frac{2A_1}{3} + A_2 + \frac{2A_3}{3} - \frac{A_5}{3} + \frac{4\omega + 1}{3}, \end{aligned}$$

and taking advantage of (6) implies

$$\begin{aligned} \frac{2N}{3}(1-q^{2N}) &= \sum_{k=1}^{2N} \frac{1}{1-q^{3k-1}} = \sum_{k=1}^{N} \frac{1}{1-q^{3k-1}} + \sum_{k=1}^{N} \frac{1}{1-q^{3(2N+1-k)-1}} \\ &= C_1 + \sum_{k=1}^{N} \frac{1}{1-q^{-3k+2}} = C_1 - \sum_{k=1}^{N} \frac{q^{3k-2}}{1-q^{3k-2}} \\ &= C_1 + N - C_2. \end{aligned}$$

The last two derivation lead to

$$C_2 = \frac{A_1}{3} + \frac{A_2}{2} + \frac{A_3}{3} - \frac{A_5}{6} + \frac{1+4\omega}{6} + \frac{N(2\omega - 1)}{6}.$$

Finally, by using (i), (ii), and (iii), we reduce equation (8) to

$$\frac{1}{6}\Big(- (A_1 + A_6) + (A_2 + A_5) - (A_3 + A_4) + N - 1 - \omega(A_2 + A_5 - (N-1))\Big) = 0$$

which holds true due to the symmetry $A_\ell + A_{7-\ell} = N - 1$, for $\ell = 1, 2$ and 3. □

5. Proof of the case $n = 2N - 1$

Let $\omega := -q^{2(2N-1)} = e^{\frac{\pi i}{3}}$ so that $\omega^2 = q^{2N-1}$, $1 - \omega + \omega^2 = 0$ and $\omega^3 = -1$.

Lemma 5.1. *We have that*

$$\omega^2 \sum_{k=1}^{N-1} \frac{q^k}{1-q^{6k}} + \sum_{k=1}^{N-1} \frac{q^{2k}}{1-q^{6k}} - \sum_{k=1}^{N} \frac{1}{1-q^{3k-2}} = -\frac{N}{3}(1+\omega). \tag{11}$$

Proof. We adopt the notations A_i from the previous section.

(i) By the partial fraction decomposition (9),

$$6q^{2N-1} \sum_{k=1}^{N-1} \frac{q^k}{1-q^{6k}} = \omega^2(A_1 - A_4 - \omega(A_3 - A_6) + \omega^2(A_5 - A_2)).$$

(ii) By the partial fraction decomposition (10),

$$6 \sum_{k=1}^{N-1} \frac{q^{2k}}{1-q^{6k}} = A_1 + A_4 - \omega(A_2 + A_5) + \omega^2(A_3 + A_6).$$

(iii) We have

$$\begin{aligned}
\sum_{k=1}^{N-1}\frac{1}{1-q^{3k-1}}+\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}} &= \sum_{k=1}^{3N-2}\frac{1}{1-q^{k}}-\sum_{k=1}^{N-1}\frac{1}{1-q^{3k}}\\
&= A_1+\sum_{k=0}^{N-1}\frac{1}{1-q^{N+k}}+A_3-\sum_{k=1}^{N-1}\frac{1}{1-q^{3k}}\\
&= A_1+\sum_{k=0}^{N-1}\frac{1}{1-q^{2N-1-k}}+A_3-\sum_{k=1}^{N-1}\frac{1}{1-q^{3k}}\\
&= A_1+\left(N-\frac{1}{1+\omega}-A_5\right)+A_3-\frac{A_1+A_3+A_5}{3}\\
&= \frac{2A_1}{3}+\frac{2A_3}{3}-\frac{4A_5}{3}+N-\frac{2-\omega}{3}
\end{aligned}$$

and invoking (6) yields

$$\begin{aligned}
\frac{2N-1}{3}(1-q^{2N-1}) = \sum_{k=1}^{2N-1}\frac{1}{1-q^{3k-1}} &= \sum_{k=1}^{N-1}\frac{1}{1-q^{3k-1}}+\sum_{k=1}^{N}\frac{1}{1-q^{3(2N-1+1-k)-1}}\\
&= \sum_{k=1}^{N-1}\frac{1}{1-q^{3k-1}}+\sum_{k=1}^{N}\frac{1}{1-q^{-3k+2}}\\
&= \sum_{k=1}^{N-1}\frac{1}{1-q^{3k-1}}-\sum_{k=1}^{N}\frac{q^{3k-2}}{1-q^{3k-2}}\\
&= \sum_{k=1}^{N-1}\frac{1}{1-q^{3k-1}}+N-\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}.
\end{aligned}$$

The last two results imply that

$$6\sum_{k=1}^{N}\frac{1}{1-q^{3k-2}}=2A_1+2A_3-4A_5+6N+\omega-2+(2N-1)(w^2-1).$$

Now, by (i), (ii), and (iii), we are able to restate (11) in terms of A_i. So, the problem boils down to exhibiting a proof for the relation

$$A_1+A_3-A_4-2A_5+A_6=0. \tag{12}$$

Since

$$\frac{x(1-x)}{1+x^3}=-\frac{2}{1+x}+\frac{1}{1-\omega x}+\frac{1}{1+\omega^2 x}$$

we have

$$\begin{aligned}
\sum_{k=1}^{N-1}\frac{q^k(1-q^k)}{1+(q^k)^3} &= A_6+A_2-2A_4,\\
\sum_{k=1}^{N-1}\frac{\omega q^k(1-\omega q^k)}{1+(\omega q^k)^3} &= A_1+A_3-2A_5,\\
\sum_{k=1}^{N-1}\frac{\omega^2 q^k(1-\omega^2 q^k)}{1+(\omega^2 q^k)^3} &= A_2+A_4-2A_6.
\end{aligned}$$

Therefore we arrive at the following equivalent form of (12):

$$\sum_{k=1}^{N-1}\left(\frac{q^k(1-q^k)}{1+(q^k)^3}+3\,\frac{\omega q^k(1-\omega q^k)}{1+(\omega q^k)^3}-\frac{\omega^2 q^k(1-\omega^2 q^k)}{1+(\omega^2 q^k)^3}\right)=0. \tag{13}$$

Since $1+(\omega q^k)^3=1-q^{3k}$ and $1+(\omega^2 q^k)^3=1+q^{3k}$, further algebraic manipulation converts (13) into

$$\sum_{k=1}^{N-1}\frac{q^k(1+\omega q^{3k})(1-\omega q^k)}{1-q^{6k}}=0. \tag{14}$$

On the other hand, we note that

$$\frac{z(1+\omega z^3)(1-\omega z)}{1-z^6}=\frac{1-\omega^2}{3}\left(\frac{1}{1-z^{-2}}+\frac{1}{1+\omega z^2}-\frac{1}{1-z^{-1}}-\frac{1}{1+\omega z}\right).$$

Letting $z=q^k$ and $\omega=-\omega^{-2}=-q^{-(2N-1)}$, our summand can be written as

$$\frac{1-\omega^3}{3}\left(\frac{1}{1-q^{-2k}}+\frac{1}{1-q^{-(2N-1-2k)}}-\frac{1}{1-q^{-k}}-\frac{1}{1-q^{-(2N-1-k)}}\right).$$

Hence, the claim now becomes

$$\sum_{k=1}^{N-1}\frac{1}{1-q^{-2k}}+\sum_{k=1}^{N-1}\frac{1}{1-q^{-(2(N-k)-1)}}=\sum_{k=1}^{N-1}\frac{1}{1-q^{-k}}+\sum_{k=1}^{N-1}\frac{1}{1-q^{-(2N-1-k)}}$$

and which in turn translates to

$$\sum_{k=1}^{N-1}\frac{1}{1-q^{-2k}}+\sum_{k=1}^{N-1}\frac{1}{1-q^{-(2k-1)}}=\sum_{k=1}^{N-1}\frac{1}{1-q^{-k}}+\sum_{k=N}^{2N-2}\frac{1}{1-q^{-k}}.$$

Indeed, equality follows here since both sides of the last equation are equal to $\sum_{k=1}^{2N-2}\frac{1}{1-q^{-k}}$. In fact, this is reminiscent of the set-theoretic identity

$$\begin{aligned}&\{k:1\leq k\leq N-1\}\cup\{2N-1-k:1\leq k\leq N-1\}\\ &=\{2k:1\leq k\leq N-1\}\cup\{2N-1-2k:1\leq k\leq N-1\}.\end{aligned}$$

The proof is complete. □

6. Conclusion

For the trigonometric functions enthusiast, the particular equation in (14) can be converted to one that involves only these circular functions. To this end, we utilize the identities

$$\frac{e^{i\theta}}{1-e^{2i\theta}}=\frac{i\csc(\theta)}{2}, \qquad \frac{1}{1+e^{2i\theta}}=\frac{1}{2}-\frac{i\tan(\theta)}{2},$$

and we suggest rewriting $\pi/6=\pi/2-(2N-1)x$ followed by replacing tan with cot via $\tan(\pi/2-t)=\cot(t)$. Here $x=\pi/(6N-3)$ and the outcome runs as

$$\begin{aligned}\frac{q^k(1+\omega q^{3k})(1-\omega q^k)}{1-q^{6k}}&=\frac{1-\omega^2}{3}\left(\frac{q^k}{1-q^{2k}}-\frac{1}{1+\omega q^k}+\frac{1}{1+\omega q^{2k}}\right)\\ &=\frac{i(1-\omega^2)}{6}\left(\csc(2kx)+\tan\left(\frac{\pi}{6}+kx\right)-\tan\left(\frac{\pi}{6}+2kx\right)\right)\\ &=\frac{i(1-\omega^2)}{6}(\csc(2kx)+\cot\left((2N-1-k)x\right)-\cot\left((2N-1-2k)x\right).\end{aligned}$$

Hence (14), reduces to verifying the trigonometric identity

$$\sum_{k=1}^{N-1}(\csc(2kx)+\cot((2N-1-k)x)-\cot((2N-1-2k)x)=0.$$

For the more number-theoretic minded reader, we present below a consequence of the identity in (14). We appreciate Terence Tao for allowing us to include his derivation in this paper. For the remainder of this section, specialize to the case where $2N-1$ is coprime to 3.

Introduce the cube root of unity $\epsilon := \omega^2 = e^{2\pi i/3} = q^{2N-1}$ where $q = e^{\frac{2\pi i}{3(2N-1)}}$. Expand the numerator in equation (14):

$$\sum_{k=1}^{N-1}\frac{q^k+\epsilon^2q^{2k}-\epsilon^2q^{4k}-\epsilon q^{5k}}{1-q^{6k}}=0.$$

From the easily verified "discrete sawtooth Fourier series" identity

$$\frac{1}{1-q^{6k}}=-\frac{1}{2N-1}\sum_{j=0}^{2N-2}jq^{6jk}$$

for any k not divisible by $2N-1$ (proven by multiplying out the denominator, cancelling terms, and applying the geometric series formula), we write the preceding identity to prove as

$$\sum_{j=0}^{2N-2}j\sum_{k=1}^{N-1}(q^{(6j+1)k}+\epsilon^2q^{(6j+2)k}-\epsilon^2q^{(6j+4)k}-\epsilon q^{(6j+5)k})=0.$$

Since $\gcd(2N-1,3)=1$, we can write $q=\epsilon^{2N-1}\zeta$ for some primitive $(2N-1)^{\text{th}}$ root ζ of unity. We then reduce to

$$\sum_{j=0}^{2N-2}j\sum_{k=1}^{N-1}\left(\epsilon^{(2N-1)k}\zeta^{(6j+1)k}+\epsilon^{2(2N-1)k+2}\zeta^{(6j+2)k}\right)$$
$$=\sum_{j=0}^{2N-2}j\sum_{k=1}^{N-1}\left(\epsilon^{(2N-1)k+2}\zeta^{(6j+4)k}+\epsilon^{2(2N-1)k+1}\zeta^{(6j+5)k}\right).$$

From Galois theory we can see that the net coefficient of ζ^a would have to be independent of a for each primitive residue class a mod $2N-1$. We summarize this discussion in the next declaration.

Corollary 6.1. *Let $N>1$ be a natural number such that $2N-1$ is not divisible by 3, let $\chi:\mathbb{Z}\to\mathbb{C}$ be a non-principal Dirichlet character of period $2N-1$, and let $\epsilon := e^{2\pi i/3}$. Then, we have the character sum identity*

$$\left(\sum_{j=0}^{2N-2}j\cdot\chi(6j+1)\right)\left(\sum_{k=1}^{2N-2}\epsilon^{(2N-1)k}\cdot\chi(k)\right) \tag{15}$$
$$=-\left(\sum_{j=0}^{2N-2}j\cdot\chi(6j+2)\right)\left(\sum_{k=1-N}^{N-1}\epsilon^{2(2N-1)k+2}\cdot\chi(k)\right).$$

We conclude with a problem proposed by Terence Tao.

Question. *Is there a direct proof of the identity* (15) *that does not rely on* (14)*?*

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