MAXIMUM PRINCIPLES FOR A CLASS OF NONLINEAR SECOND ORDER ELLIPTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we investigate maximum principles for functionals defined on solutions to special partial differential equations of elliptic type, extending results by Payne and Philippin. We apply such maximum principles to investigate one overdetermined problem.

1. Introduction.

We consider classical solutions $u = u(x)$ of the quasilinear second order equation

$$\big(g(q^2)u_i\big)_i = h(q^2) \tag{1.1}$$

in domains $\Omega \subset R^N$. Here and in the sequel the subindex i $(i = 1, ..., N)$ denotes partial differentiation with respect to x^i, the summation convention (from 1 to N) over repeated indices is in effect, $q^2 = u_i u_i$, g and h are two smooth functions. In order for equation (1.1) to be elliptic we suppose $g > 0$ and $G > 0$, where

$$G(\xi) = g(\xi) + 2\xi g'(\xi). \tag{1.2}$$

Following Payne and Philippin [5,6,7] we derive some maximum principles for functionals $\Phi(u, q)$ defined on solutions u of equation (1.1). Of course, there are infinitely many choices for such functionals. In order to exploit the corresponding maximum principle for getting more information on u, not only the functional must satisfy a maximum principle, but, in addition, there must exist some domain Ω and some solution u of (1.1) for which $\Phi(u, q)$ is a constant throughout Ω. Such maximum principles are named "best possible" maximum principles ([4]). For applications of such "best possible" inequalities in fluid mechanics, geometry and in other areas we refer to [4].

In [6] Payne and Philippin consider the functional

$$\Phi(u, q) = \frac{1}{2}\int_0^{q^2} \frac{G(\xi)}{h(\xi)} d\xi - u \tag{1.3}$$

and prove that, if $u(x)$ satisfies the equation (1.1) then $\Phi(u, q)$ assumes its maximum value either on the boundary of Ω or when $q = 0$. If $u = u(x^1)$ is a function

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depending on one variable only and if it is a solution of (1.1) then the corresponding $\Phi(u,q)$ is a constant. In the same paper [6], Payne and Philippin define

$$\Psi(u,q) = \frac{N}{2}\int_0^{q^2} \frac{G(\xi)}{h(\xi)} d\xi - u \tag{1.4}$$

and prove that, if $\Psi(u,q)$ is computed on any solution of equation (1.1) then it assumes its maximum value on the boundary of Ω. In case h is a nonvanishing constant and $u = u(r)$ is the radial solution of equation (1.1) satisfying $u'(0) = 0$, then $\Psi(u,q)$ is a constant. These results have been extended to more general equations in [7].

In Section 2 of this paper we exhibit a new class of functionals which satisfy "best possible" maximum principles. These functionals are expressed in terms of solutions to an ordinary differential equation related to (1.1).

In Section 3 we consider the equation:

$$\left(g(q^2)u_i\right)_i = N, \tag{1.5}$$

where g satisfies suitable hypotheses. Assume equation (1.5) has a smooth solution $u(x)$ in a convex ringshaped domain $\Omega \subset R^N$ bounded externally by a (hyper) surface Γ_0 and internally by a (hyper) surface Γ_1. We show that, if such a solution satisfies the following (overdetermined) boundary conditions

$$u|_{\Gamma_0} = 0, \quad u|_{\Gamma_1} = -c_1,$$
$$u_n|_{\Gamma_0} = q_0, \quad u_n|_{\Gamma_1} = 0,$$

where c_1 and q_0 are positive free constants, then Γ_0 and Γ_1 must be two concentric N-spheres. Similar problems have been investigated by several authors. In [9] Philippin and Payne discussed the equation:

$$\left(q^{N-2}u_i\right)_i = 0$$

under the boundary conditions

$$u|_{\Gamma_0} = 0, \quad u|_{\Gamma_1} = -1, \tag{1.6}$$

$$u_n|_{\Gamma_0} = q_0, \quad u_n|_{\Gamma_1} = -q_1, \tag{1.7}$$

where q_0 and q_1 are free constants. They proved that if this problem is solvable then Ω must be radially symmetric. In [8] Philippin solved tha same problem in case the equation is $\Delta u = 0$ and the boundary conditions are (1.6), (1.7). In [10] Porru and Ragnedda investigated the above problem when the equation is

$$\left(q^{p-2}u_i\right)_i = 0,$$

$p > 1$, again under conditions (1.6), (1.7). The case when Ω is a bounded simple connected domain has been studied by Serrin in [12]. By using the moving plane method he has found that if $u(x)$ is a smooth solution of equation (1.1) and satisfies

$$u|_{\partial\Omega} = 0, \quad u_n|_{\partial\Omega} = q_0, \tag{1.8}$$

$(q_0 \neq 0)$ then Ω must be a sphere. The same result has been found by Weinberger [13] for the special case $\Delta u = -1$ by using a different method. Extending Weinberger's method, Garofalo and Lewis [1] have solved the overdetermined problem (1.5), (1.8) allowing $u(x)$ to be a generalized solution.

2. Maximum principles.

Let us prove first some preliminary lemmas.

Lemma 2.1. *Let $z(t)$ be either a strictly convex or a strictly concave C^2 function in (t_0, t_1) and let $\psi(s)$ be the inverse of $z'(t)$. Then the function of t*

$$\int_{z'(t_0)}^{z'(t)} s\psi'(s)\,ds - z(t)$$

is a constant on (t_0, t_1).

Proof. The proof is trivial. If we replace s by $z'(\tau)$ in the above integral we obtain

$$\int_{z'(t_0)}^{z'(t)} s\psi'(s)\,ds - z(t) = \int_{t_0}^{t} z'(\tau)\,d\tau - z(t) = -z(t_0).$$

The lemma is proved.

Lemma 2.2. *Let $u = u(x)$ be a smooth function satisfying $\nabla u \neq 0$ in $\Omega \subset R^N$, and let $\psi = \psi(s)$ be a smooth function in $(0, \infty)$. Then we have in Ω*

$$\begin{aligned} &\psi^2 u_{ih} u_{ih} \\ &\geq \big(\psi^2 - (\psi')^2 q^2\big) q_i q_i + \big(2\psi' q - 2\psi\big) q_i u_i - N q^2 + 2\psi q \Delta u, \end{aligned} \tag{2.1}$$

where $q = |\nabla u|$ and $\psi = \psi(q)$. Furthermore, equality holds in (2.1) throughout Ω if and only if $\psi^2(q) = |x - x_0|^2$.

Proof. Let δ^{ih} be the Kronecker delta. By

$$\sum_{i,h}^{1,N} \Big(\big(\frac{\psi(q)}{q} u_i\big)_h - \delta^{ih} \Big)^2 \geq 0 \tag{2.2}$$

it follows

$$\sum_{i,h}^{1,N} \Big(\frac{\psi}{q} u_{ih} + \frac{\psi'}{q} q_h u_i - \frac{\psi}{q^2} q_h u_i - \delta^{ih} \Big)^2 \geq 0.$$

Easy computations give

$$\frac{\psi^2}{q^2} u_{ih} u_{ih} - \big(\frac{\psi^2}{q^2} - (\psi')^2\big) q_i q_i - \big(2\frac{\psi'}{q} - 2\frac{\psi}{q^2}\big) q_i u_i + N - \frac{2\psi}{q} \Delta u \geq 0,$$

where the identities $u_i u_i = q^2$, $u_{ih} u_h = q q_i$ have been used. Inequality (2.1) follows. Of course, we have equality in (2.1) if and only if equality holds in (2.2), that is, if and only if

$$\big(\frac{\psi(q)}{q} u_i\big)_h = \delta^{ih}, \quad i, h = 1, \dots, N.$$

By solving the last equations we obtain

$$\frac{\psi(q)}{q} u_i = x^i - x_0^i, \quad i = 1, \dots, N. \tag{2.3}$$

Since $u_i u_i = q^2$, these equalities imply

$$\psi^2(q) = |x - x_0|^2.$$

The lemma has been proved.

Corollary. *If we have equality in* (2.1) *and if* $\psi(s)$ *is either strictly increasing or strictly decreasing then* $u(x)$ *must be a radial function.*

Proof. Since $\psi^2(q) = |x - x_0|^2$, by (2.3) we find:

$$\nabla u = H(|x - x_0|)(x - x_0),$$

where H is an appropriate function of one variable only. The result follows.

Let us come to equation (1.1). The function g is assumed to be smooth on $(0, \infty)$ and to satisfy

$$g(\xi) > 0, \quad \lim_{\xi \to 0} g(\xi^2)\xi = 0, \quad \lim_{\xi \to \infty} g(\xi^2)\xi = \infty, \quad G(\xi) > 0, \tag{2.4}$$

where $G(\xi)$ is defined as in (1.2). The function h is supposed to be smooth in $[0, \infty)$ and to satisfy

$$h(\xi) > 0, \quad h'(\xi) \geq 0 \quad \forall \xi \geq 0. \tag{2.5}$$

The case $h(\xi) < 0$, $h'(\xi) \leq 0$ can be reduced to the case in above by changing u with $-u$ in (1.1).

Define the ordinary differential equation

$$\left(t^{N-1} g(\phi^2)\phi\right)' = t^{N-1} h(\phi^2). \tag{2.6}$$

Observe that, if $u(r)$, $r = |x|$, is a radial solution of equation (1.1) for $r_1 < r < r_2$ then $\phi(t) = u'(t)$ is a solution of (2.6) for $r_1 < t < r_2$.

Lemma 2.3. *Assume conditions* (2.4), (2.5). *Given* $t_0 \geq 0$, *let* $\phi(t)$ *be the solution of equation* (2.6) *satisfying* $\phi(t_0) = 0$. *If* (t_0, t_1) *is the maximal interval of existence for* $\phi(t)$ *then* $\phi'(t) > 0$ *on* (t_0, t_1) *and* $\phi(t) \to \infty$ *as* $t \to t_1$. *Here* t_1 *may be finite or* ∞.

Proof. This lemma is probably known, but we give a proof for completeness. From the equation (2.6) and the condition $\phi(t_0) = 0$ one finds $\phi'(t) > 0$ on the (maximal) interval (t_0, a), with $a \leq t_1$. We claim that $a = t_1$. By contradiction, let $a < t_1$, so $\phi'(a) = 0$. Since h is nondecreasing, the function $h(\phi^2(t))$ is nondecreasing on (t_0, a). Hence, integration of (2.6) on (t_0, t), $t \leq a$, yields

$$t^{N-1} g(\phi^2)\phi \leq \frac{t^N - t_0^N}{N} h(\phi^2) \leq \frac{t^N}{N} h(\phi^2). \tag{2.7}$$

Insertion of (2.7) into (2.6) rewritten as $G(\phi^2)\phi' + \frac{N-1}{t} g(\phi^2)\phi = h(\phi^2)$ leads to

$$G(\phi^2)\phi' \geq \frac{1}{N} h(\phi^2). \tag{2.8}$$

At $t = a$, (2.8) implies $\phi'(a) > 0$, which contradicts the assumption $\phi'(a) = 0$. Hence $\phi(t)$ is strictly increasing on (t_0, t_1).

If t_1 is finite then $\phi(t) \to \infty$ as $t \to t_1$ because of the maximality of the interval (t_0, t_1). Let $t_1 = \infty$. For $t \geq t_0$, (2.6) implies

$$\Big(t^{N-1} g(\phi^2)\phi\Big)' \geq t^{N-1} h(0).$$

Integrating over (t_0, t) we find

$$g(\phi^2)\phi \geq t\Big(1 - \Big(\frac{t_0}{t}\Big)^N\Big)\frac{h(0)}{N}.$$

Taking into account conditions (2.4), the above inequality implies that $\phi(t) \to \infty$ as $t \to \infty$. The lemma is proved.

Observe that equation (1.1) may be rewritten as

$$\Delta u + f(q) u_i u_j u_{ij} = k(q), \tag{2.9}$$

where $f(q) = 2g'(q^2)/g(q^2)$ and $k(q) = h(q^2)/g(q^2)$. The ordinary differential equation (2.6) in terms of f and k reads as

$$\phi'\big(1 + \phi^2 f(\phi)\big) + \frac{N-1}{t}\phi = k(\phi). \tag{2.10}$$

Theorem 2.1. *Assume conditions (2.4), (2.5). Given $t_0 \geq 0$, let $\phi(t)$ be the solution of equation (2.10) satisfying $\phi(t_0) = 0$, and let $\psi(s)$ be the inverse function of $\phi(t)$. If $u(x)$ is a solution of equation (2.9) such that $\nabla u \neq 0$ in Ω then the function*

$$\Phi(u, q) = \int_0^{q(x)} s\psi'(s)\, ds - u(x) \tag{2.11}$$

assumes its maximum value on the boundary of Ω. Moreover, $\Phi(u, q)$ is a constant if $u(x)$ is the (radially symmetric) solution $u(x) = F(|x|)$, $F'(t) = \phi(t)$.

Proof. By Lemma 2.3, $\phi(t)$ is strictly increasing, hence the second part of the theorem follows by Lemma 2.1 when $z'(t) = \phi(t)$. For proving the first part we put

$$v(x) = \int_0^{q(x)} s\psi'(s)\, ds - u(x),$$

where $u(x)$ is a solution of equation (2.9). We have

$$v_i = \psi'(q) q q_i - u_i, \quad i = 1, \ldots, N. \tag{2.12}$$

By (2.12) we obtain

$$v_{ij} = \psi'(q_i q_j + q q_{ij}) + \psi'' q q_i q_j - u_{ij}, \quad i, j = 1, \ldots, N.$$

From the identities $qq_i = u_{ih}u_h$ we get

$$q_iq_j + qq_{ij} = u_{ih}u_{jh} + u_{ijh}u_h.$$

Consequently, we find

$$v_{ij} = \psi'(u_{ih}u_{jh} + u_{ijh}u_h) + \psi'' qq_iq_j - u_{ij}. \tag{2.13}$$

Let us define

$$a^{ij} = \delta^{ij} + f(q)u_iu_j, \quad i,j = 1,\dots,N, \tag{2.14}$$

where δ^{ij} is the Kronecker delta. In virtue of conditions (2.4) the matrix $[a^{ij}]$ is positive definite. By using (2.13) and (2.14) we find

$$\frac{1}{\psi'}a^{ij}v_{ij} = u_{ih}u_{ih} + fq^2q_iq_i$$

$$+a^{ij}u_{ijh}u_h + \frac{\psi''}{\psi'}q(q_iq_i + fq_iu_iq_ju_j) - \frac{k}{\psi'},$$

where the equation (2.9) rewritten as $a^{ij}u_{ij} = k$ has been used. Easy computations yield

$$a^{ij}u_{ijh} = (a^{ij}u_{ij})_h - (a^{ij})_h u_{ij} = k'q_h - f'q_hqq_iu_i - 2fu_{ih}qq_i,$$

where $f' = f'(q)$ and $k' = k'(q)$. Hence

$$\frac{1}{\psi'}a^{ij}v_{ij} \tag{2.15}$$

$$= u_{ih}u_{ih} - fq^2q_iq_i + k'q_iu_i - f'qq_iu_iq_ju_j + \frac{\psi''}{\psi'}q(q_iq_i + fq_iu_iq_ju_j) - \frac{k}{\psi'}.$$

Equality (2.15) and inequality (2.1) give

$$\frac{1}{\psi'}a^{ij}v_{ij} \tag{2.16}$$

$$\geq \Big(1 - \frac{(\psi')^2}{\psi^2}q^2\Big)q_iq_i + \Big(2\frac{\psi'}{\psi^2}q - \frac{2}{\psi}\Big)q_iu_i - \frac{N}{\psi^2}q^2 + \frac{2}{\psi}kq - \frac{2}{\psi}fq^2q_iu_i$$

$$-fq^2q_iq_i + k'q_iu_i - f'qq_iu_iq_ju_j + \frac{\psi''}{\psi'}q(q_iq_i + fq_iu_iq_ju_j) - \frac{k}{\psi'},$$

where the equation $\Delta u = k - fqq_iu_i$ has been used. By (2.12) we obtain

$$q_ju_j = \frac{v_ju_j}{\psi'q} + \frac{q}{\psi'}, \quad q_iq_i = \Big(q_i + \frac{u_i}{\psi'q}\Big)\frac{v_i}{\psi'q} + \frac{1}{(\psi')^2}. \tag{2.17}$$

Insertion of (2.17) into (2.16) yields:

$$\begin{aligned}\frac{1}{\psi'}a^{ij}v_{ij}\\ \geq \frac{1}{(\psi')^2}+\frac{q^2}{\psi^2}(1-N)-\frac{2q}{\psi\psi'}(1+fq^2)+\frac{2kq}{\psi}+\frac{k'q}{\psi'}-\\ -\frac{f'q^3}{(\psi')^2}-\frac{fq^2}{(\psi')^2}+\frac{\psi''}{(\psi')^3}q(1+fq^2)-\frac{k}{\psi'}-\frac{1}{\psi'}b^iv_i,\end{aligned}\tag{2.18}$$

where b^i (i=1,...,N) is a regular vector field (recall that, by assumption, $q(x)>0$). From equation (2.10) with $t=\psi(q)$ (and, consequently, $\phi(t)=q$) we find

$$k=\frac{1}{\psi'}(1+fq^2)+\frac{N-1}{\psi}q.\tag{2.19}$$

By using equation (2.19), inequality (2.18) becomes:

$$\begin{aligned}\frac{1}{\psi'}a^{ij}v_{ij}+\frac{1}{\psi'}b^iv_i\\ \geq \frac{\psi''}{(\psi')^3}q(1+fq^2)-\frac{2fq^2}{(\psi')^2}-\frac{f'q^3}{(\psi')^2}+\frac{k'q}{\psi'}-\frac{N-1}{\psi\psi'}q+\frac{N-1}{\psi^2}q^2.\end{aligned}\tag{2.20}$$

Differentiation with respect to t in equation (2.10) yields:

$$\phi''(1+\phi^2f)+2(\phi')^2\phi f+(\phi')^2\phi^2f'+\frac{N-1}{t}\phi'-\frac{N-1}{t^2}\phi=k'\phi'.\tag{2.21}$$

Since $\psi(q)$ is the inverse of $\phi(t)$, we have

$$\phi'(t)=\frac{1}{\psi'(q)},\quad \phi''(t)=-\frac{\psi''(q)}{(\psi'(q))^3}.$$

Hence, the equation (2.21) may be rewritten as

$$\frac{k'}{\psi'}=-\frac{\psi''}{(\psi')^3}(1+fq^2)+\frac{2fq}{(\psi')^2}+\frac{f'q^2}{(\psi')^2}+\frac{N-1}{\psi\psi'}-\frac{N-1}{\psi^2}q.\tag{2.22}$$

Insertion of (2.22) into (2.20) yields

$$a^{ij}v_{ij}+b^iv_i\geq 0.\tag{2.23}$$

The theorem follows by (2.23) and the classical maximum principle [11,2].

Remark 2.1 If equality holds in (2.23) then equality holds in (2.1) and, by the corollary to Lemma 2.2, $u(x)$ is a radial function.

Remark 2.2 If h is a (positive) constant and if $t_0=0$ then the functional (2.11) is the same as that defined in (1.4).

In case of dimension two, Theorem 2.1 can be improved. In fact, we have the following

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, if N=2 the function $v(x) = \Phi(u, q)$ defined in* (2.11) *assumes its maximum value and its minimum value on the boundary of* Ω.

Proof. The proof of this theorem is the same as that of Theorem 2.1 up to equation (2.15). At this point, instead of using inequality (2.1), we make use of the following equality (true for $N = 2$ only, ([7]) p. 43))

$$u_{ih}u_{ih} = (\Delta u)^2 + 2q_iq_i - 2\Delta u q_i \frac{u_i}{q}. \tag{2.24}$$

Since $\Delta u = k - fqq_iu_i$, equality (2.24) yields

$$u_{ih}u_{ih} = \big(k - fqq_iu_i\big)^2 + 2q_iq_i - 2kq_i\frac{u_i}{q} + 2fq_iu_iq_ju_j. \tag{2.25}$$

Insertion of (2.25) into (2.15) and use of equations (2.17) lead to

$$\frac{1}{\psi'}a^{ij}v_{ij} \tag{2.26}$$

$$= \big(k - \frac{fq^2}{\psi'}\big)^2 + \frac{2}{(\psi')^2} - \frac{3k}{\psi'} + \frac{fq^2}{(\psi')^2}$$

$$+\frac{k'q}{\psi'} - \frac{f'q^3}{(\psi')^2} + \frac{\psi''q}{(\psi')^3}(1 + fq^2) - \frac{1}{\psi'}d^iv_i,$$

where d^i (i=1, 2) is a regular vector field. By equation (2.19) with $N = 2$ we have

$$k = \frac{1}{\psi'} + \frac{1}{\psi'}fq^2 + \frac{1}{\psi}q. \tag{2.27}$$

By (2.22) we have

$$\frac{k'}{\psi'} = -\frac{\psi''}{(\psi')^3}(1 + fq^2) + \frac{2fq}{(\psi')^2} + \frac{f'q^2}{(\psi')^2} + \frac{1}{\psi\psi'} - \frac{q}{\psi^2}. \tag{2.28}$$

Insertion of (2.27) and (2.28) into (2.26) leads to

$$a^{ij}v_{ij} + d^iv_i = 0.$$

The theorem follows by classical maximum principles [11,2].

3. An overdetermined problem.

Throughout this section we assume $\Omega \subset R^N$ to be a smooth ringshaped domain bounded externally by a (hyper) surface Γ_0 and internally by a (hyper) surface Γ_1. We also suppose Γ_0 and Γ_1 enclose convex domains Ω_0 and Ω_1, respectively. In Ω we investigate the following (overdetermined) problem:

$$\big(g(q^2)u_i\big)_i = N, \tag{3.1}$$

$$u|_{\Gamma_0} = 0, \quad u|_{\Gamma_1} = -c_1, \tag{3.2}$$

$$u_n|_{\Gamma_0} = q_0, \quad u_n|_{\Gamma_1} = 0, \tag{3.3}$$

where c_1 and q_0 are two positive free constants and the subindex n denotes normal external differentiation. The function $g(\xi)$ is assumed to satisfy conditions (2.4). According to (1.2) we have

$$G(s^2) = \frac{d}{ds}\big(g(s^2)s\big). \tag{3.4}$$

We are interested only in smooth solutions of equation (3.1) whose gradient is nonvanishing in Ω.

Theorem 3.1. *If* (2.4) *holds then problem* (3.1), (3.2), (3.3) *is solvable if and only if* q_0, c_1 *satisfy*

$$\frac{1}{N}\int_0^{q_0} sG(s^2)ds < c_1 < \int_0^{q_0} sG(s^2)ds \tag{3.5}$$

and , $_0$, , $_1$ *are two suitable concentric N-spheres.*

Proof. The ordinary differential equation (2.6) corresponding to our partial differential equation (3.1) is

$$\Big(t^{N-1}g(\phi^2)\phi\Big)' = \big(t^N\big)'. \tag{3.6}$$

Take $\alpha \geq 0$ and assume $\phi(\alpha) = 0$. Integrating (3.6) over (α, t) we obtain

$$g(\phi^2)\phi = t(1 - \alpha^N t^{-N}).$$

Taking into account conditions (2.4) one concludes that $\phi(t)$ is defined on (α, ∞). If $\psi(s)$ denotes the inverse function of $\phi(t)$ then by the last equation we have

$$g(s^2)s = (1 - \alpha^N\psi^{-N})\psi. \tag{3.7}$$

From (3.7) we get

$$\psi' = \frac{G(s^2)\psi^N}{\psi^N + (N-1)\alpha^N}, \quad \psi(0) = \alpha. \tag{3.8}$$

Suppose problem (3.1), (3.2), (3.3) has a regular solution $u = u(x)$. Let us consider the function $\psi(s)$ defined by (3.7) when $\alpha = 0$. We find $\psi(s) = g(s^2)s$ and $\psi'(s) = G(s^2)$. By Theorem 2.1, the function

$$v(x) = \int_0^{q(x)} sG(s^2)ds - u(x) \tag{3.9}$$

attains its maximum value on $\Gamma_1 \cup \Gamma_0$. On Γ_1 and on Γ_0 we have

$$v_n = qG(q^2)q_n - u_n, \tag{3.10}$$

where the subindex n, as before, denotes normal external differentiation. Equation (3.1) rewritten in normal coordinates reads as

$$G(q^2)q_n + g(q^2)(N-1)Ku_n = N,$$

where K is the mean curvature of the corresponding level surface. Since Γ_1 is smooth and since $u_n = -q = 0$ on Γ_1, we find

$$G(q^2)q_n = N \quad \text{on } \Gamma_1. \tag{3.11}$$

It follows that v_n vanishes at each point in Γ_1. Hence, by Hopf's second principle, $v(x)$ cannot take its maximum value on Γ_1 unless it is a constant in Ω. Consequently, such a maximum value is attained in Γ_0. Using conditions (3.2), (3.3) and comparing the values of v on Γ_1 and Γ_0 we find

$$c_1 \leq \int_0^{q_0} sG(s^2)ds. \tag{3.12}$$

Using again conditions (3.2), (3.3) we obtain

$$v(x) \leq \int_0^{q_0} sG(s^2)ds,$$

from which it follows

$$\int_\Omega v(x)dx \leq (|\Omega_0| - |\Omega_1|) \int_0^{q_0} sG(s^2)ds. \tag{3.13}$$

On the other hand, from equation (3.1) we find

$$\frac{1}{N}\Big(x^j u_j g u_i - gq^2 x^i + (N-1)ugu_i\Big)_i = \frac{x^i}{N}\Big(gqq_i - \big(gq^2\big)_i\Big) + x^i u_i + (N-1)u. \tag{3.14}$$

By using Green's formula as well as conditions (3.2), (3.3) we obtain

$$\int_\Omega \Big(x^j u_j g u_i - gq^2 x^i + (N-1)ugu_i\Big)_i dx = 0. \tag{3.15}$$

Since

$$gqq_i - \big(gq^2\big)_i = -(gq)_i q = -\Big(\int_0^q sG(s^2)ds\Big)_i$$

we find

$$\int_\Omega \frac{x^i}{N}\Big(gqq_i - \big(gq^2\big)_i\Big)dx = -\int_\Omega \frac{x^i}{N}\Big(\int_0^q sG(s^2)ds\Big)_i dx.$$

By using again Green's formula, the boundary conditions (3.3) and the well known equation

$$\int_{\Gamma_0} x^i n^i ds = N|\Omega_0|$$

we obtain

$$\int_\Omega \frac{x^i}{N}\Big(gqq_i - \big(gq^2\big)_i\Big)dx = -\int_0^{q_0} sG(s^2)ds|\Omega_0| + \int_\Omega\Big(\int_0^q sG(s^2)ds\Big)dx. \tag{3.16}$$

By using once more Green's formula and the boundary conditions (3.2) we find

$$\int_\Omega x^i u_i dx = -c_1\int_{\Gamma_1} x^i n^i ds - N\int_\Omega u\, dx.$$

Since

$$\int_{\Gamma_1} x^i n^i ds = -N|\Omega_1|,$$

the previous equation gives

$$\int_\Omega x^i u_i dx = c_1 N|\Omega_1| - N\int_\Omega u\, dx. \tag{3.17}$$

Integration in (3.14) and use of (3.15), (3.16), (3.17) and (3.9) lead to

$$\int_\Omega v(x)dx = \int_0^{q_0} sG(s^2)ds|\Omega_0| - c_1 N|\Omega_1|. \tag{3.18}$$

By (3.13) and (3.18) it follows

$$\frac{1}{N}\int_0^{q_0} sG(s^2)ds \le c_1. \tag{3.19}$$

If equality were to occur in (3.12) then equality would occur in (3.13) and in (3.19), a contradiction. Hence we must have strict inequality in (3.12) and in (3.19). Inequalities (3.5) have been proved.

Now let us consider the function $\psi(s)$ defined by (3.7) for $\alpha > 0$, and discuss the following equation

$$\int_0^{q_0} s\psi'(s)\, ds = c_1. \tag{3.20}$$

By (3.7) with $\alpha = 0$ we find $s\psi'(s) = sG(s^2)$. Since $\psi(0) = \alpha$ and since $\psi(s)$ is increasing, for s fixed, $\psi \to \infty$ as $\alpha \to \infty$. As a consequence, by (3.7) we infer that when $\alpha \to \infty$ then $\frac{\alpha}{\psi(s)} \to 1$. Hence, by (3.8) it follows that

$$s\psi'(s) \to \frac{1}{N} sG(s^2) \;\text{ as }\; \alpha \to \infty.$$

Therefore, since $s\psi'(s)$ decreases as α increases, and since q_0 and c_1 satisfy inequalities (3.5), there is a unique positive α which solves equation (3.20). By using this value of α let us define

$$v(x) = \int_0^{q(x)} s\psi'(s)\,ds - u(x). \tag{3.21}$$

In virtue of conditions (3.2), (3.3) and (3.20), the function $v(x)$ assumes the same value on Γ_0 and on Γ_1. If $N = 2$ then Theorem 2.2 implies that $v(x)$ is a constant in Ω. For general N, let us compute the normal derivative of v on Γ_1. We find

$$v_n = q\psi'(q)q_n - u_n. \tag{3.22}$$

By using (3.8) with $s = q$ we have

$$\psi' q_n = \frac{\psi^N}{\psi^N + (N-1)\alpha^N} G(q^2) q_n. \tag{3.23}$$

By (3.23) and (3.11), recalling that $\psi(0) = \alpha$ we find that $\psi'(q)q_n = 1$ on Γ_1. Hence v_n vanishes on Γ_1. Since $v(x)$ satisfies the elliptic inequality (2.23), by Hopf's second principle $v(x)$ must be a constant in Ω. Because $v(x)$ is a constant we have equality in (2.23). Then, by Remark 2.1, $u(x)$ must be a radial function. Consequently, taking into account conditions (3.2), Γ_0 and Γ_1 must be N-spheres.

Now suppose q_0 and c_1 satisfy (3.5). Let Γ_1 be the N-sphere whose radius r_1 is equal to the value of α which solves equation (3.20), and let Γ_0 be the N-sphere concentric with Γ_1 whose radius r_0 satisfies the following equation:

$$g(q_0^2)q_0 r_0^{N-1} = r_0^N - r_1^N. \tag{3.24}$$

The function $\psi(s)$ defined by

$$g(s^2)s = (1 - r_1^N \psi^{-N})\psi \tag{3.25}$$

is strictly increasing in $(0,\infty)$ and satisfies the conditions $\psi(0) = r_1$ and (by (3.24)) $\psi(q_0) = r_0$. Its inverse function $s = \phi(r)$ satisfies equation (3.6) with $t = r$. The theorem has been proved.

If $N = 2$ and $g(q^2) = q^{p-2}$, $p > 1$, then equation (3.1) is related to the torsion problem [3,5]. In this case, conditions (3.5) read as

$$\frac{1}{2}\Big(1 - \frac{1}{p}\Big) q_0^p < c_1 < \Big(1 - \frac{1}{p}\Big) q_0^p.$$

Equation (3.7) becomes

$$s^{p-1}\psi = \psi^2 - \alpha^2, \tag{3.28}$$

from which we find

$$\psi = \frac{1}{2}\Big(s^{p-1} + \big(s^{2p-2} + 4\alpha^2\big)^{\frac{1}{2}}\Big).$$

The radius r_1 of the circle $,_1$ is the solution of the equation

$$\frac{p-1}{2}\int_0^{q_0}\Big(s^{p-1}+s^{2p-2}\big(s^{2p-2}+4r_1^2\big)^{-\frac{1}{2}}\Big)\,ds=c_1.$$

The radius r_0 of the circle $,_0$ is given by

$$r_0=\frac{1}{2}\Big(q_0^{p-1}+\big(q_0^{2p-2}+4r_1^2\big)^{\frac{1}{2}}\Big).$$

The solution $u(r)$ is

$$u(r)=\int_{r_0}^{r}\Big(t-r_1^2t^{-1}\Big)^{\frac{1}{p-1}}\,dt.$$

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