q-APÉRY IRRATIONALITY PROOFS BY q-WZ PAIRS

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Abstract. Using WZ pairs, Apéry-style proofs of the irrationality of the q-analogues of the Harmonic series and $Ln(2)$ are given. For the q-analogue of $Ln(2)$, this method produces an improved irrationality measure.

0. Introduction:

Let us define the following q-analogues of the Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and Ln(2), respectively by:

$$h_q(1) := \sum_{k=1}^{\infty} \frac{1}{q^k - 1} \qquad (\text{for } |q| > 1), \tag{0.1}$$

$$Ln_q(2) := \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n - 1} \qquad (\text{for } |q| \neq 0, 1). \tag{0.2}$$

In 1948, Paul Erdös [E1] proved the irrationality of $h_2(1)$. Recently, Peter Borwein used Padé approximation techniques [B1] and some complex analysis methods [B2] to prove the irrationality of both $h_q(1)$ and $Ln_q(2)$. Here we present a proof in the spirit of Apéry's magnificent proof of the irrationality of $\zeta(3)$ [A], which was later delightfully accounted by Alf van der Poorten [P]. This method of proof gives favorable irrationality measure (=4.80) for $Ln_q(2)$ campared to the irrationality measure (=54.0) implied in [B1], [B2]. Further discussion of irrationality results for certain series is to be found in Erdös [E2].

We will assume familiarity with ref. [Z]. In particular,
$\binom{n}{k}_q := \frac{(q)_n}{(q)_k (q)_{n-k}}$, where $(q)_0 := 1$ and $(q)_n := (1-q)\cdots(1-q^n)$, for $n \geq 1$.

N and K are *forward shift operators* on n and k, respectively.

$\Delta_n := N - 1$, $\Delta_k := K - 1$.

A pair $(F(n,k), G(n,k))$ of discrete functions is called a *q-WZ pair* if:
1. NF/F, KF/F, NG/G and KG/G are all rational functions of q^n and q^k, and
2. $\Delta_n F = \Delta_k G$.

Given such a pair (F, G), then $\omega = F(n,k)\delta k + G(n,k)\delta n$ is called a *q-WZ 1-form.*

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1. A scheme for proving the irrationality of the *q-harmonic series* $h_q(1)$**:**

The claims made in subsections **1.1-1.5** below were found using the Maple Package `qEKHAD` accompanying [PWZ]. The relevant script substantiating our claims can be found in this paper's Web Pages.

1.1. The q-WZ 1-form ω is:

$$\omega = \frac{-1}{\binom{n+k+1}{k}_q (q)_{n+1}} \{\delta k + \frac{q^{n+1}}{(q^{n+1}-1)} \delta n\}.$$

1.2. The choice of the potential $c(n,k)$ is:

$$c(n,k) = \sum_{m=1}^{n} \frac{q^m}{(1-q^m)(q)_m} + \sum_{m=1}^{k} \frac{1}{(q^m-1)} \frac{1}{\binom{n+m}{m}_q (q)_n}.$$

1.3. The choice of the mollifier $b(n,k)$ is:

$$b(n,k) = (-1)^k q^{k(k+1)/2} \binom{n+k}{k}_q \binom{n}{k}_q .$$

1.4. We define two sequences:

$$a(n) = \sum_{k=0}^{n} c(n,k) b(n,k), \qquad \text{and} \qquad b(n) = \sum_{k=0}^{n} b(n,k).$$

1.5. Introduce $L = y_2(n)N^2 + y_1(n)N + y_0(n)$ and $B(n,k) = P_q^1(n,k) b(n+1,k)$, where

$$A(n,k) = c(n,k)B(n,k) + \frac{(-1)^k q^{2n+3}}{q^{n+1}-1} \binom{n+1}{k}_q \frac{q^{\binom{k}{2}}}{(q)_{n+2}} P_q^2(n,k) \qquad \text{and}$$

$$P_q^1(n,k) = -q\alpha_n^2 \beta_k^{-1}(q^2\alpha_n + 2q) + q\alpha_n^2(q^2\alpha_n^3 + 2q(q+1)\alpha_n^2 + 3q\alpha_n - (q+1) - (\alpha_n+2)\beta_k)$$

$$P_q^2(n,k) = q^2\alpha_n^2 + q\alpha_n - 2 + \beta_k(q^2\alpha_n^5 + q(2q+1)\alpha_n^4 - 2\ alpha_n^3 - \alpha_n^3\beta_k - (2-q^{-1})\alpha_n^2\beta_k$$
$$- (3q+5)\alpha_n^2 + 2q^{-1}\alpha_n\beta_k + (q-1+2q^{-1})\alpha_n + (1+3q^{-1})),$$

$y_0(n) = q(\alpha_n - 1)(q\alpha_n + 2)$, $y_2(n) = (q\alpha_n - 1)(\alpha_n + 2)$, $\alpha_n = q^{n+1}$, $\beta_k = q^{k+1}$ and

$y_1(n) = q^3\alpha_n^5 + 2q^2(q+1)\alpha_n^4 + q^2\alpha_n^3 - 4q(q+1)\alpha_n^2 + (q^2-4q+1)\alpha_n + 2(q+1).$

Then

$$(*) \qquad L(b(n,k)) = B(n,k) - B(n,k-1) \qquad \text{and} \qquad L(b(n,k)c(n,k)) = A(n,k) - A(n,k-1).$$

Now, summing over k in $(*)$ shows that both sequences $a(n)$ and $b(n)$ are solutions of $Lu(n) = 0$.

1.6. Set $b_n = b(n)$ and $a_n = a(n)$. Now, since $b_{n+1} > b_n$ and $Lb_n = 0$, that is, $y_2(n)b_{n+2} + y_1(n)b_{n+1} + y_0(n)b_n = 0$, then asymptotically we have that

$$\frac{b_{n+2}}{b_{n+1}} = O\left(\frac{y_1(n)}{y_2(n)}\right) = O\left(q^{3n+3}\right).$$

Hence,

$$b_n = O\left(q^{\frac{3n^2}{2}}\right). \tag{1.6.1}$$

On the other hand, $La_n = 0$ and $Lb_n = 0$ lead to the system of recurrence relations,

$$y_2(n)a_{n+2} + y_1(n)a_{n+1} + y_0(n)a_n = 0, \qquad y_2(n)b_{n+2} + y_1(n)b_{n+1} + y_0(n)b_n = 0. \tag{1.6.2}$$

Multiplying out the first and the second equations in (1.6.2), respectively by b_{n+2} and a_{n+2}, and subtracting we obtain

$$y_1(n)(a_{n+1}b_{n+2} - b_{n+1}a_{n+2}) = y_0(n)(a_{n+2}b_n - b_{n+2}a_n).$$

Rewriting this in the form

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_{n+2}}{b_{n+2}} = \frac{y_0(n)}{y_1(n)}\frac{b_n}{b_{n+1}}\left(\frac{a_{n+2}}{b_{n+2}} - \frac{a_n}{b_n}\right)$$

leads to the estimate

$$\left|\frac{a_{n+1}}{b_{n+1}} - \frac{a_{n+2}}{b_{n+2}}\right| \le \left|\frac{y_0(n)}{y_1(n)}\frac{b_n}{b_{n+1}}\left(\frac{a_{n+2}}{b_{n+2}} - \frac{a_{n+1}}{b_{n+1}}\right)\right| + \left|\frac{y_0(n)}{y_1(n)}\frac{b_n}{b_{n+1}}\left(\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n}\right)\right|,$$

which in turn yields

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = O\left(b_n^{-2}\right). \tag{1.6.3}$$

Therefore,

$$h_q(1) - \frac{a_n}{b_n} = O\left(b_n^{-2}\right). \tag{1.6.4}$$

In particular, the sequence of rational numbers $\frac{a_n}{b_n}$ converges moderately quickly to $h_q(1)$.

1.7. For a given prime p, let ord_pk denote the exponent of p in the prime expansion of k. Then we observe that

$$ord_p \binom{n}{m}_q \leq ord_p(q)_n - ord_p(q)_m. \tag{1.7.1}$$

Note:

$$\binom{n+k}{k}_q \binom{k}{m}_q = \binom{n+m}{m}_q \binom{n+k}{k-m}_q. \tag{1.7.2}$$

Lemma 1: The sequences

$$u_n = a_n(q)_{n+1} \prod_{s=[n/2]}^{n} (1-q^s) \qquad \text{and} \qquad z_n = b_n(q)_{n+1} \prod_{s=[n/2]}^{n} (1-q^s)$$

are polynomials in q with integer coefficients, and moreover

$$z_n = O\left(q^{19n^2/8}\right). \tag{1.7.3}$$

Proof: Applying (1.7.1) and (1.7.2), we can estimate the denominator of u_n as:

$$\begin{aligned} ord_p \left(\frac{(q^m-1)(q)_n \binom{n+m}{m}_q}{\binom{n+k}{k}_q} \right) &\leq ord_p \left(\frac{(q^m-1)(q)_n \binom{k}{m}_q}{\binom{n+k}{k-m}_q} \right) \\ &\leq ord_p(q)_n + ord_p(q^m-1) + ord_p(q)_k - ord_p(q)_m \\ &\leq ord_p(q)_n + ord_p \prod_{s=[n/2]}^{n} (1-q^s) + ord_p(q)_k - ord_p(q)_m \\ &\leq ord_p \left((q)_n \prod_{s=[n/2]}^{n} (1-q^s) \right), \end{aligned}$$

since $m \leq k \leq n$. This proves the claim on u_n. And (1.7.3) follows from (1.6.1). The rest is trivial.

Lemma 2: $h_q(1)$ - $\frac{u_n}{z_n} = O\left(\frac{1}{z_n^{1+\delta}}\right)$; where $\delta = 0.26316\ldots > 0$.

proof: From (1.6.1), (1.6.4) and (1.7.3), we gather that

$$h_q(1) - \frac{u_n}{z_n} = O\left(b_n^{-2}\right) = O\left(q^{-3n^2}\right) = O\left(z_n^{-1-(5/19)}\right).$$

Thus, we have proved:

Theorem 1: If $|q| > 1$ is an integer, $h_q(1)$ is irrational with irrationality measure 4.80.

Remark 1: By invoking Theorem 7 ([Z], p.596) with ω as in 1.1, we obtain the series acceleration:

$$h_q(1) = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q)_n} \qquad \text{and} \qquad h_q(1) = \sum_{n=1}^{\infty} \frac{1-q^n-q^{2n}}{(q^n-1)\binom{2n}{n}_q (q)_n}.$$

2. A scheme for proving the irrationality of $Ln_q(2)$:

The claims made in subsections **2.1-2.5** below were found using the Maple Package **qEKHAD** accompanying [PWZ]. The relevant script substantiating our claims can be found in this paper's Web Pages.

2.1. The qWZ 1-form ω is:

$$\omega = \frac{(-1)^k}{(1-q^{k+1})} \frac{(q)_n}{\binom{n+k+1}{k+1}_q (q^2)_n} \{\delta k + \frac{q^{n+1}}{(1+q^{n+1})} \delta n\}.$$

2.2. The choice of the potential $c(n,k)$ is:

$$c(n,k) = \sum_{m=1}^{n} \frac{q^m (q)_m}{(1-q^m)(q^2)_m} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{(1-q^m)} \frac{(q)_n}{\binom{n+m}{m}_q (q^2)_n}.$$

2.3. The choice of the mollifier $b(n,k)$ is:

$$b(n,k) = q^{k(k+1)/2} \binom{n+k}{k}_q \binom{n}{k}_q.$$

2.4. We define two sequences:

$$a(n) = \sum_{k=0}^{n} c(n,k) b(n,k), \qquad \text{and} \qquad b(n) = \sum_{k=0}^{n} b(n,k).$$

2.5. Introduce $L = y_2(n)N^2 + y_1(n)N + y_0(n)$ and $B(n,k) = P_q^1(n,k) b(n+1,k)$, where

$$A(n,k) = c(n,k)B(n,k) + \frac{(-1)^k q^{2n+3}}{1-q^{n+1}} \binom{n+1}{k}_q \frac{q^{\binom{k}{2}} (q)_{n+1}}{(q^2)_{n+1}} P_q^2(n,k) \qquad \text{and}$$

$$P_q^1(n,k) = q\alpha_n^2 \left[q^3\alpha_n^5 + q^2(1+q)\alpha_n^4 + 2q(1+q^2)\alpha_n^3 - (1-q+q^2)\alpha_n - 3(1+q)\right]$$
$$+ q\alpha_n^2 \left[q\beta_k^{-1}(q^2\alpha_n^3 + q(1+q)\alpha_n^2 + (2-q)\alpha_n - 2) + (q\alpha_n^3 + (q-1)\alpha_n^2 + (2q-1)\alpha_n)\alpha_k - 2\right]$$

$$P_q^2(n,k) = q^2\alpha_n^3 + q(1+q)\alpha_n^2 + (2+q)\alpha_n + 2 - \alpha_n\alpha_k^2\left[\alpha_n^3 + (1-q^{-1})\alpha_n^2 + (2-q)\alpha_n - 2q^{-1}\right]$$
$$- \alpha_k\left[q^2\alpha_n^6 + q(1+q)\alpha_n^5 + (2+q+2q^2)\alpha_n^4 + (1+q)\alpha_n^3 + 2\alpha_n^2 - (2+q+q^{-1})\alpha_n + (q^{-1}-1)\right],$$

$$y_0(n) = -q(\alpha_n - 1)(\alpha_n + 1)(q^2\alpha_n^2 + q\alpha_n + 2), \quad y_2(n) = -(q\alpha_n - 1)(q\alpha_n + 1)(\alpha_n^2 + \alpha_n + 2),$$

$$y_1(n) = q^4\alpha_n^7 + q^2(1+q)(q\alpha_n^6 + \alpha_n^4) + q(1+q+q^2)(2q\alpha_n^5 + alpha_n^3) - (1+3q+3q^2+q^3)\alpha_n^2 - (1+q^2)(2+\alpha_n),$$

and $\alpha_n = q^{n+1}, \ \beta_k = q^{k+1}$.

Then

$$(**) \qquad L(b(n,k)) = B(n,k) - B(n,k-1) \qquad \text{and} \qquad L(b(n,k)c(n,k)) = A(n,k) - A(n,k-1).$$

Now, summing over k in $(**)$ shows that both sequences $a(n)$ and $b(n)$ are solutions of $Lu(n) = 0$.

2.6. Similar arguments and estimates as in (1.6) above lead to

$$Ln_q(2) - \frac{a_n}{b_n} = O\left(b_n^{-2}\right). \tag{2.6.1}$$

In particular, the sequence of rational numbers $\frac{a_n}{b_n}$ converges moderately quickly to $Ln_q(2)$.

2.7. Lemma 3: The sequences

$$v_n = a_n \prod_{t=1}^{n}(1+q^t) \prod_{s=[n/2]}^{n}(1-q^s) \qquad \text{and} \qquad w_n = b_n \prod_{t=1}^{n}(1+q^t) \prod_{s=[n/2]}^{n}(1-q^s)$$

are polynomials in q with integer coefficients, and moreover

$$w_n = O\left(q^{19n^2/8}\right). \tag{2.7.1}$$

Proof: Applying (1.7.1) and (1.7.2), we have estimates for the denominator of v_n:

$$\begin{aligned} ord_p\left(\frac{(1-q^m)(q^2)_n \binom{n+m}{m}_q}{\binom{n+k}{k}_q (q)_n}\right) &\le ord_p\left(\frac{(q^m-1)(q^2)_n \binom{k}{m}_q}{\binom{n+k}{k-m}_q (q)_n}\right) \\ &\le ord_p\left(\frac{(q^2)_n}{(q)_n}\right) + ord_p(q^m-1) + ord_p(q)_k - ord_p(q)_m \\ &\le ord_p\left(\frac{(q^2)_n}{(q)_n}\right) + ord_p \prod_{s=[n/2]}^{n}(1-q^s) + ord_p(q)_k - ord_p(q)_m \\ &\le ord_p\left(\prod_{t=1}^{n}(1+q^t)\prod_{s=[n/2]}^{n}(1-q^s)\right), \end{aligned}$$

since $m \le k \le n$. This proves the claim on v_n. And (2.7.1) follows from (1.6.1). The rest is trivial.

Lemma 4: $Ln_q(2)$ - $\frac{v_n}{w_n} = O\left(\frac{1}{w_n^{1+\delta}}\right)$; where $\delta = 0.26316\ldots > 0$.

proof: Combining (1.6.1), (2.6.1) and (2.7.1), we find that

$$Ln_q(2) - \frac{v_n}{w_n} = O\left(b_n^{-2}\right) = O\left(q^{-3n^2}\right) = O\left(w_n^{-1-(5/19)}\right).$$

Thus, we have proved:

Theorem 2: If $|q| \neq 0, 1$ is an integer, $Ln_q(2)$ is irrational with irrationality measure 4.80.

Remark 2: We invoke Theorem 7 ([Z], p. 596) with ω as in 2.1, to get the accelerated series:

$$Ln_q(2) = \sum_{n=1}^{\infty} \frac{q^n (q)_n}{(1-q^n)(q^2)_n} \qquad \text{and} \qquad Ln_q(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(q)_n(1-q^{3n})}{(1-q^n)^2 \binom{2n}{n}_q (q^2)_n}.$$

References

[A] R. Apéry, *Irrationalitè de $\zeta(2)$ et $\zeta(3)$*, Asterisque **61** (1979), 11-13.

[B1] P. Borwein, *On the irrationality of $\sum 1/(q^n + r)$*, J. Number Theory **37** (1991), 253-259.

[B2] P. Borwein, *On the irrationality of certain series*, Proc. Camb. Phil. Soc. **112** (1992), 141-146.

[E1] P. Erdös, *On arithmetical properties of Lambert Series*, J. Indian Math. Soc. (N.S.) **12** (1948), 63-66.

[E2] P. Erdös, *On the irrationality of certain series: problems and results*, In New Advances in Transcendence Theory (Cambridge University Press), 1988, pp. 102-109.

[P] A. van der Poorten, *A proof that Euler missed ..., Apéry's proof of the irrationality of $\zeta(3)$*, Math. Intel. **1** (1979), 195-203.

[PWZ] M. Petkovšek, H.S. Wilf, D. Zeilberger, *"A=B"*, A.K. Peters Ltd., 1996.

`The package qEKHAD is available by the www at http://www.math.temple.edu/~zeilberg/programs.html`

[Z] D. Zeilberger, *Closed Form (pun intended!)*, Contemporary Mathematics **143** (1993), 579-607.