Q-ANALOGUES TO SOME DETERMINANTS IN THE R.B. AND C.K. PAPERS

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Consider the determinants defined by

$$M_n(a,b;x) := \det_{0\le i,j\le n-1}\left(\binom{2(i+a)+2(j+b)+x}{i+a} - \binom{2(i+a)+2(j+b)+x}{i+a-1}\right).$$

REMARK: Clearly, $M_n(a,b;x) = M_n(a,0;x+2b)$.

The following are found in [1] and [2]: For each integer $n\ge 1$, $M_n(0,0;x) = 2^{\binom{n}{2}}$ and $M_n(1,0;x) = 2^{\binom{n+1}{2}}\binom{n+\frac{x-1}{2}}{n}$. It follows immediately that $M_n(0,1;x) = 2^{\binom{n}{2}}$ and $M_n(1,1;x) = 2^{\binom{n+1}{2}}\binom{n+\frac{x+1}{2}}{n}$.

We will now record immediate q-analogues to the above determinants, as follows:

$$_qM_n(a,b;x) := \det_{0\le i,j\le n-1}\left(\binom{2(i+a)+2(j+b)+x}{i+a}_q - \binom{2(i+a)+2(j+b)+x}{i+a-1}_q\right).$$

Claim: For each integer $n\ge 1$, we have that

$$_qM_n(0,0;x) = q^{n_1(x)}\prod_{j=1}^{n-1}(1+q^j)^{n-j} = q^{n_1(x)}\frac{(n-1)!!_{q^2}}{(n-1)!!_q}, \qquad \text{and}$$

$$_qM_n(1,0;x) = q^{n_2(x)}\prod_{j=1}^{n-1}(1+q^j)^{n-j}\prod_{i=1}^{n}\frac{1-q^{2j+x-1}}{1-q^j} = q^{n_2(x)}\binom{n+\frac{x-1}{2}}{n}_{q^2}\frac{n!!_{q^2}}{n!!_q};$$

where $n_1(x) = \frac{1}{4}\binom{2n}{3} + (x+n-1)\binom{n}{2}$ and $n_2(x) = 5\binom{n+1}{3} + \binom{n}{1} + x\binom{n}{2}$.

References

[1] R. Bacher, *Determinants of matrices related to the Pascal Triangle*, J. Théorie Nombres Bordeaux, 14 (2002).

[2] C. Krattethaler, *Evaluations of some determinants of matrices related to the Pascal Triangle*, Séminaire Lotharingien de Combin., 47 (2002), Article B47g.