SOLUTION TO MONTHLY PROBLEM PROPOSED BY D. E. KNUTH

Tewodros Amdeberhan

Department of Mathematics, Temple University, Philadelphia PA 19122, USA
tewodros@euclid.math.temple.edu

Problem #10568: **[P]** Let n be a nonnegative integer. The sequence defined by $x_0 = n$ and $x_{k+1} = x_k - \lceil\sqrt{x_k}\rceil$ for $k \geq 0$ converges to 0. Let $f(n)$ be the number of steps required; i.e. $x_{f(n)} = 0$ but $x_{f(n)-1} > 0$. Find a closed form for $f(n)$.

Solution: Consider the following partition of the positive integers into "intervals":
for each nonnegative integer m, define successively

$$\{[a_m(k)+1, b_m(k)], [b_m(k)+1, a_m(k+1)]\},$$

for $k = 1, 2, \ldots, 2^m - 1$, and augmented by $[a_m(2^m)+1, b_m(2^m)]$; where

$$a_m(k) := 4^m + (2k-3)2^m + k(k-1), \qquad \text{and} \qquad b_m(k) := 4^m + (2k-2)2^m + k^2.$$

Claim:

$$f(n) = \begin{cases} 2^{m+1} - m - 2 + 2k - 2, & \text{if n} \in [a_m(k)+1, b_m(k)] \\ 2^{m+1} - m - 2 + 2k - 1, & \text{if n} \in [b_m(k)+1, a_m(k+1)] \end{cases}$$

To this end it suffices to verify the inequalities:

$$a_m(k) \leq b_m(k) - \lceil\sqrt{b}_m(k)\rceil < a_m(k) + 1, \tag{1}$$

and

$$b_m(k) \leq a_m(k+1) - \lceil\sqrt{a}_m(k+1)\rceil < b_m(k) + 1. \tag{2}$$

These follow respectively from

$$2^m + k - 1 < \lceil\sqrt{b}_m(k)\rceil \leq 2^m + k, \tag{3}$$

and

$$2^m + k - 1 < \lceil\sqrt{a}_m(k+1)\rceil \leq 2^m + k. \tag{4}$$

Then, the formula in the claim is a consequence of successive counting of the intervals. □

References:

[P] P 10568, *American Mathematical Monthly,* **(104) #1,** 1997.

Typeset by $\mathcal{AMS}$-TEX