SOLUTION TO PROBLEM #10651
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PROBLEM: [**P**] If u_1 and u_2 are nonconstant real functions of two variables, and if u_1, u_2, and u_1u_2 are all harmonic in a simply connected domain D, prove that $u_2 = av_1 + b$, where v_1 is a harmonic conjugate of u_1 in D, and a and b are real constants.

PROOF: In R^2, we write w_x and w_y for $\partial w/\partial x$ and $\partial w/\partial y$, respectively. Let $f = u_1 + iv_1$. Then f^2 is analytic, and hence $2u_1v_1 = Im(f^2)$ is harmonic.

Using the assumptions and the equations,

$$\Delta(u_1u_2) = \Delta(u_1) + \Delta(u_2) + 2\nabla(u_1)\cdot\nabla(u_2), \qquad \Delta(u_1v_1) = \Delta(u_1) + \Delta(v_1) + 2\nabla(u_1)\cdot\nabla(v_1)$$

it follows that both vectors ∇u_2 and ∇v_1 are orthogonal to ∇u_1, in R^2. Thus

$$\nabla u_2 = a\nabla v_1, \tag{1}$$

for some function $a = a(x,y)$. Consequently, $\Delta u_2 = a\Delta v_1 + \nabla a \cdot \nabla v_1$. Therefore, we obtain

$$\nabla v_1 \cdot (a_x, a_y) = 0. \tag{2}$$

Rewriting equation (1) as: $(u_2)_x = a(v_1)_x, (u_2)_y = a(v_1)_y$ and differentiating in y and x, respectively gives

$$(u_2)_{xy} = a_y(v_1)_x + a(v_1)_{xy}, \qquad (u_2)_{yx} = a_x(v_1)_y + a(v_1)_{yx}.$$

This shows that

$$\nabla v_1 \cdot (a_y, -a_x) = 0. \tag{3}$$

Combining equations (2) and (3), we have $\nabla a \equiv 0$, i.e. a is a constant. This in turn implies that $\nabla(u_2 - av_1) = \nabla u_2 - a\nabla v_1 \equiv 0$, proving that $u_2 - av_1$ is a constant and completing the proof. □

References:

[**P**] P #10651, *American Mathematical Monthly,* **(105) #3,** 1998.