SOLUTION TO PROBLEM #10697
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PROBLEM: [**P**] Given n distinct nonzero complex numbers $z_1, z_2, \ldots, z_n$, show that

$$\sum_{k=1}^{n} \frac{1}{z_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{z_k - z_j} = \frac{(-1)^{n+1}}{z_1 z_2 \cdots z_n}. \tag{1}$$

PROOF: Multiply both sides of (1) by $(-1)^{n-1} z_1 z_2 \cdots z_n$, and get

$$\sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{z_j}{z_j - z_k} = 1. \tag{2}$$

Now, we proceed by induction on n. This is trivial for $n = 1$, with an empty product in (2). So, assume equation (2) holds for n and verify it for $n + 1$.

$$\begin{aligned}\sum_{k=1}^{n+1} \prod_{\substack{j=1 \\ j\neq k}}^{n+1} \frac{z_j}{z_j - z_k} &= \sum_{k=1}^{n} \frac{z_{n+1}}{z_{n+1} - z_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{z_j}{z_j - z_k} + \prod_{j=1}^{n} \frac{z_j}{z_j - z_{n+1}} \\ &= \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{z_j}{z_j - z_k} + \sum_{k=1}^{n} \frac{z_k}{z_{n+1} - z_k} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{z_j}{z_j - z_k} + \prod_{j=1}^{n} \frac{z_j}{z_j - z_{n+1}}. \end{aligned} \tag{3}$$

If we could prove that the last two terms add up to 0, the proof is complete. However, this is equivalent to:

$$\sum_{k=1}^{n} \frac{z_k}{z_k - z_{n+1}} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{z_j}{z_j - z_k} = \prod_{j=1}^{n} \frac{z_j}{z_j - z_{n+1}}. \tag{4}$$

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Dividing out through by the right-hand term in (4), this in turn takes the form:

$$\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{z_j - z_{n+1}}{z_j - z_k} = 1.$$

Letting $\alpha_j = z_j - z_{n+1}$, the last equation immediately follows from the induction assumption, i.e.

$$\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{\alpha_j}{\alpha_j - \alpha_k} = 1 \qquad \square.$$

References:

[**P**] P #10697, *American Mathematical Monthly,* **(105) #10,** December 1998.