SOLUTION TO PROBLEM #10714
PROPOSED BY JET WIMP

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PROBLEM: [P] For $a \in (-\pi/2, \pi/2)$, define

$$c_n(t) := \frac{1}{e^{at}\cos a}\left(\frac{d}{da}\right)^n \left(e^{at}\cos a\right)$$

for every nonnegative integer n, so that $c_n(t)$ is a monic polynomial of degree n. Let G_n denote the $(n+1)$-by-$(n+1)$ determinant $|c_{j+k}(t)|_{j,k=0,1,\ldots,n}$. Evaluate $G_n(t)$.

SOLUTION: Let $D^n = \left(\frac{d}{da}\right)^n$ and $z_\pm = e^{a(t\pm i)}$. So, we have $e^{at}\cos a = (z_+ + z_-)/2$ and $D^n\left(e^{at}\cos a\right) = ((t+i)^n z_+ + (t-i)^n z_-)/2$. Since $z_\pm$ are solutions of the differential equation

$$D^2 y - 2tDy + (1+t^2)y = 0, \tag{1}$$

so are the functions $D^n\left(e^{at}\cos a\right)$. Thus clearly, equation (1) implies the recurrence

$$c_{n+2}(t) - 2tc_{n+1}(t) + (1+t^2)c_n(t) \equiv 0, \qquad \text{for } n \geq 0.$$

It follows that in the Hankel matrix $\left(c_{j+k}(t)\right)_{j,k=0,1,\ldots,n}$, any three consecutive columns are linearly dependant, for $n \geq 2$. Therefore, we get

$$G_n(t) = 0 \qquad \text{for all } n \geq 2; \qquad \text{while} \qquad G_0(t) = 1 \qquad G_1(t) = -\sec^2 a. \square$$

References:

[P] P #10714, *American Mathematical Monthly,* **(106) #2,** February 1999.