SOLUTION TO PROBLEM #10723
PROPOSED BY C. J. HILLAR

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Proposed by C. J. Hillar, Yale Univ., New Haven, CT Let p be an odd prime. Prove that $\sum_{i=1}^{p-1} 2^i \cdot i^{p-2} \equiv \sum_{i=1}^{(p-1)/2} i^{p-2} (\bmod p)$.

*Solution by T. Amdeberhan, DeVry Institute, North Brunswick, NJ*Assmue all congruences are $(\bmod p)$, here. In the group $Z_p^* := \{1, \dots, p-1\}$, we have $k^{p-2} \equiv k^{-1}$ (Fermat's Little Theorem) and also $\binom{p-1}{j} \equiv (-1)^j$, since $\binom{p-1}{j} = j^{-1}(p-j)\binom{p-1}{j-1} \equiv -\binom{p-1}{j-1}$ and $\binom{p-1}{1} \equiv -1$. Thus,

$$\sum_{k=j}^{p-1} \binom{k}{j} k^{-1} \equiv j^{-1} \sum_{k=j}^{p-1} \binom{k-1}{j-1} = j^{-1} \binom{p-1}{j} \equiv j^{-1}(-1)^j. \tag{1}$$

Let $f(x) := \sum_{k=1}^{p-1} x^k k^{-1}$, expand $(1-x)^k$ and employ (1) to obtain

$$f(1-x) = \sum_{k=1}^{p-1} k^{-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j x^j \equiv \sum_{j=1}^{p-1} (-1)^j x^j \sum_{k=j}^{p-1} \binom{k}{j} k^{-1} \equiv f(x);$$

where $\sum_{k=1}^{p-1} k^{-1} \equiv \sum_{n=1}^{p-1} n \equiv 0$ has been used prior to switching summation indices.

So, $f(2) = \sum_{k=1}^{p-1} 2^k \cdot k^{-1} \equiv f(-1) = \sum_{k=1}^{p-1} (-1)^k k^{-1} = \sum_{\text{k even}}^{1,p-1} k^{-1} - \sum_{\text{k odd}}^{1,p-1} k^{-1}$. As p is odd, $p-k$ is even whenever k is odd and hence

$$\begin{aligned} \sum_{k=1}^{p-1} 2^k \cdot k^{-1} &\equiv \sum_{\text{k even}}^{1,p-1} k^{-1} - \sum_{\text{k odd}}^{1,p-1} k^{-1} \\ &\equiv \sum_{\text{k even}}^{1,p-1} k^{-1} + \sum_{\text{k odd}}^{1,p-1} (p-k)^{-1} \equiv 2 \sum_{\text{k even}}^{1,p-1} k^{-1} \\ &= 2 \sum_{k=1}^{(p-1)/2} (2k)^{-1} \equiv \sum_{k=1}^{(p-1)/2} k^{-1}. \square \end{aligned}$$

References:

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