SOLUTION TO PROBLEM #10739
PROPOSED BY OSCAR CIAURRI

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Proposed by Oscar Caurri, Lograño, Spain. Suppose that $f:[0,1]\to R$ has a continuous second derivative with $f''(x)>0$ on $(0,1)$, and suppose that $f(0)=0$. Choose $a\in(0,1)$, such that $f'(a)<f(1)$. Show that there is a unique $b\in(a,1)$ such that $f'(a)=\frac{f(b)}{b}$.

Solution by T. Amdeberhan, DeVry Institute, North Brunswick, NJ. By the Mean Value Theorem for derivatives, there exits $c\in(0,a)$ such that $f'(c)=\frac{f(a)}{a}$. $f''(x)>0$ implies that $f'(x)$ is 1-1 and increasing, therefore $f'(c)<f'(a)$. It follows that

$$\frac{f(a)}{a}<f'(a). \tag{1}$$

Define $g(x):=f(x)-xf'(a)$. The choice of a, yields $g(1)=f(1)-f'(a)>0$. But $g(a)=f(a)-af'(a)<0$ by (1). Since $g(x)$ is clearly continuous, the Intermediate Value Theorem asserts the existence of $b\in(a,1)$ such that $g(b)=0$, i.e. $f'(a)=\frac{f(b)}{b}$.

Now, since $f'(x)$ is strictly increasing and $g(x)$ is differentiable, we have $g'(x)=f'(x)-f'(a)>0$, for $x>a$. Thus $g(x)$ is one-to-one, thereby proving the uniqueness of b. $\square$

References:

[**P**] P #10739, *American Mathematical Monthly,* **(106) #6,** June-July 1999.

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