SOLUTION TO PROBLEM #10744
PROPOSED BY PETER LINDQUIST, AND ET. AL

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Proposed by Peter Lindquist, Norwegian Univ. of Sci. and Tech., Trondheim, Norway, and Jaak Peetre, Univ. of Lund, Lund, Sweden. Fix $p > 0$, and define functions, $S(x)$, $C(x)$ and $T(x)$ for sufficiently small x by

$$x = \int_0^{S(x)} \frac{dt}{(1-t^p)^{\frac{p-1}{p}}}, \qquad x = \int_{C(x)}^{1} \frac{dt}{(1-t^p)^{\frac{p-1}{p}}}, \qquad x = \int_0^{T(x)} \frac{dt}{(1+t^p)^{\frac{2}{p}}}.$$

Show that $S^p(x) + C^p(x) = 1$ and that $T(x) = S(x)/C(x)$. The case $p = 2$ yields the familiar trigonometric formulas.

Solution by T. Amdeberhan, DeVry Institute, North Brunswick, NJ. Assuming $x \geq 0$, we have $S(x), T(x) \geq 0$ and $C(x) \leq 1$. Since the integrands are positive and continuous, we are ensured of monotonicity, invertibility and differentiability. Moreover, we have

$$S'(x) = (1 - S^p(x))^{\frac{p-1}{p}}, \qquad S(0) = 0 \tag{1}$$

$$-C'(x) = (1 - C^p(x)^{\frac{p-1}{p}}, \qquad C(0) = 1 \tag{2}$$

$$T'(x) = (1 + T^p(x))^{\frac{2}{p}}, \qquad T(0) = 0. \tag{3}$$

To verify the first assertion, it suffices to show that $y = (1 - C^p)^{1/p}$ satisfies the same 1st order initial value problem (1), as $S(x)$. This follows from $y(0) = 0$ and

$$y' = \frac{1}{p}(1-C^p)^{\frac{1-p}{p}}(pC^{p-1})(-C') = \frac{1}{p}(1-C^p)^{\frac{1-p}{p}}(pC^{p-1})(1-C^p)^{\frac{p-1}{p}} = C^{p-1} = (1-y^p)^{\frac{p-1}{p}}.$$

By uniqueness of solutions, we get $y(x) = S(x)$. To prove the second part of the assertion, note that $S'(x) = C^{p-1}(x)$ and also $C'(x) = -S^{p-1}(x)$. We want to show that $z = S/C$ meets the same IVP in (3) as does T. Since $z(0) = 0$, and

$$z' = \left(\frac{S}{C}\right)' = \frac{S'C - C'S}{C^2} = \frac{C^p + S^p}{C^2} = \frac{1}{C^2}, \qquad (1+z^p)^{\frac{2}{p}} = \left(\frac{C^p + S^p}{C^p}\right)^{\frac{2}{p}} = \frac{1}{C^2},$$

we conclude that $T = z = S/C$. □

References:

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