SOLUTION TO PROBLEM #10762
PROPOSED BY LEROY QUET

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Proposed by Leroy Quet, Denver, CO. Let $x_1 = 1$, and for $m \geq 1$ let $x_{m+1} = (m+3/2)^{-1} \sum_{k=1}^{m} x_k x_{m+1-k}$. Evaluate $\lim_{m \to \infty} x_m / x_{m+1}$.

Solution by T. Amdeberhan, DeVry Institute, North Brunswick, NJ. Define the formal series $f(a) := \sum_{m=1}^{\infty} x_m a^m$. After multiplying the above recurrence by y^{2m+2} and summing over $m \geq 1$, we obtain

$$\sum_{m=1}^{\infty} (2m+3) x_{m+1} y^{2m+2} = 2 \sum_{m=1}^{\infty} y^{2m+2} \sum_{k=1}^{m} x_k x_{m+1-k}. \tag{1}$$

Applying Cauchy's product and derivatives, equation (1) takes the form

$$-3x_1 y^2 + \frac{d}{dy}\left(y \sum_{m=1}^{\infty} x_m y^{2m} \right) = 2 \left(\sum_{k=1}^{\infty} x_k y^{2k} \right)^2.$$

Using $x_1 = 1$ and the definition of f, we obtain the differential equation

$$2y^2 f'(y^2) + f(y^2) - 3y^2 = 2f(y^2)^2, \qquad f(0) = 0. \tag{2}$$

Then, we get the solution

$$g(y) := f(y^2) = \frac{1}{2} - \frac{y\sqrt{6}}{2} cot(y\sqrt{6}).$$

Clearly, $g(y)$ has radius of convergence $\pi/\sqrt{6}$ about $y = 0$ and hence $f(a)$ has $\pi^2/6$. But this implies that

$$\lim_{m \to \infty} \frac{x_m}{x_{m+1}} = \frac{\pi^2}{6}. \qquad \square$$

References:

[**P**] P #10762, *American Mathematical Monthly,* **(106) #9,** November 1999.