SOLUTION TO PROBLEM #10791
PROPOSED BY A. FEKETE

DeVry Institute, Mathematics
630 US Highway One, North Brunswick, NJ 08902
amdberhan@admin.nj.devry.edu

Proposed by Antal Fekete, Memorial University of Newfoundland, St. John's, NF, Canada. Show that

$$\left(\sum_{i=0}^{\infty}\frac{(2i+1)^n}{(2i+1)!}\right)^2-\left(\sum_{i=0}^{\infty}\frac{(2i)^n}{(2i)!}\right)^2 \qquad \text{and} \qquad \left(\sum_{i=0}^{\infty}(-1)^i\frac{(2i+1)^n}{(2i+1)!}\right)^2+\left(\sum_{i=0}^{\infty}(-1)^i\frac{(2i)^n}{(2i)!}\right)^2$$

are integers for every nonnegative integer n.

Solution by Tewodros Amdeberhan, DeVry Institute, North Brunswick, NJ. Consider the two bases $V=(1,x,x(x-1),x(x-1)(x-2),\dots,x(x-1)\cdots(x-n+1))$ and $U=(1,x,x^2,\dots,x^n)$ for the set of polynomials of degree n or less. In particular, the change of bases transformation matrices M and M^{-1} (note that $Det(M)=1$),

$$V^T=M\cdot U^T \qquad \text{and} \qquad U^T=M^{-1}\cdot V^T,$$

between U and V, are obviously of integer entries. Therefore, the coefficients $a_k(n)$ are integrals in

$$x^n=\sum_{k=0}^{n}a_k(n)\frac{x!}{(x-k)!}, \qquad \text{for integers } n\geq 0.$$

Replacing x by $2i+1$, we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{(2i+1)^n}{(2i+1)!}&=\sum_{i=0}^{\infty}\frac{1}{(2i+1)!}\sum_{k=0}^{n}a_k(n)\frac{(2i+1)!}{(2i+1-k)!}\\ &=\sum_{k\text{even}}^{0,n}a_k(n)\sum_{2i+1\geq k}^{\infty}\frac{1}{(2i+1-k)!}+\sum_{k\text{odd}}^{0,n}a_k(n)\sum_{2i+1\geq k}^{\infty}\frac{1}{(2i+1-k)!}.\end{aligned}$$

Thus, $\sum_{i=0}^{\infty}\frac{(2i+1)^n}{(2i+1)!}=A_n\cdot sinh(1)+B_n\cdot cosh(1)$ and $\sum_{i=0}^{\infty}\frac{(2i)^n}{(2i)!}=A_n\cdot cosh(1)+B_n\cdot sinh(1)$, where $A_n:=\sum_{k\text{even}}^{0,n}a_k(n)$ and $B_n:=\sum_{k\text{odd}}^{0,n}a_k(n)$. Consequently,

$$(1) \qquad \left(\sum_{i=0}^{\infty}\frac{(2i+1)^n}{(2i+1)!}\right)^2-\left(\sum_{i=0}^{\infty}\frac{(2i)^n}{(2i)!}\right)^2=B_n^2-A_n^2.$$

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Analogously,

$$\sum_{i=0}^{\infty}(-1)^i\frac{(2i+1)^n}{(2i+1)!}=\sum_{k\text{even}}^{0,n}a_k(n)\sum_{2i+1\geq k}^{\infty}\frac{(-1)^i}{(2i+1-k)!}+\sum_{k\text{odd}}^{0,n}a_k(n)\sum_{2i+1\geq k}^{\infty}\frac{(-1)^i}{(2i+1-k)!}$$
$$=\sum_{k\text{even}}^{0,n}(-1)^{k/2}a_k(n)\cdot sin(1)+\sum_{k\text{odd}}^{0,n}(-1)^{(k-1)/2}a_k(n)\cdot cos(1),\qquad\text{and}$$

$$\sum_{i=0}^{\infty}(-1)^i\frac{(2i)^n}{(2i)!}=\sum_{k\text{even}}^{0,n}a_k(n)\sum_{2i\geq k}^{\infty}\frac{(-1)^i}{(2i-k)!}+\sum_{k\text{odd}}^{0,n}a_k(n)\sum_{2i\geq k}^{\infty}\frac{(-1)^i}{(2i-k)!}$$
$$=\sum_{k\text{even}}^{0,n}(-1)^{k/2}a_k(n)\cdot cos(1)+\sum_{k\text{odd}}^{0,n}(-1)^{(k+1)/2}a_k(n)\cdot sin(1).$$

Now, $\sum_{i=0}^{\infty}(-1)^i\frac{(2i+1)^n}{(2i+1)!}=C_n\cdot sin(1)+D_n\cdot cos(1)$ and $\sum_{i=0}^{\infty}(-1)^i\frac{(2i)^n}{(2i)!}=C_n\cdot cos(1)-D_n\cdot sin(1)$, where $C_n:=\sum_{k\text{even}}^{0,n}(-1)^{k/2}a_k(n)$ and $D_n:=\sum_{k\text{odd}}^{0,n}(-1)^{(k-1)/2}a_k(n)$. Consequently,

$$\left(\sum_{i=0}^{\infty}(-1)^i\frac{(2i+1)^n}{(2i+1)!}\right)^2+\left(\sum_{i=0}^{\infty}(-1)^i\frac{(2i)^n}{(2i)!}\right)^2=C_n^2+D_n^2.\qquad(2)$$

Since all the A_n, B_n, C_n and D_n are integers, the proof is complete. □

References:

[**P**] P #10791, *American Mathematical Monthly,* **(107) #3,** March 2000.