SOLUTION TO PROBLEM #10859
PROPOSED BY R. BACHER

Tewodros Amdeberhan

DeVry Institute, Mathematics
630 US Highway One, North Brunswick, NJ 08902
amdberhan@admin.nj.devry.edu

Proposed by Roland Bacher, Institute Fourier, St. Martin d'Héres, France. Compute the determinant of the n-by-n matrix whose i,j–entry is $q^{(i-j)^2}$ for $0 \le i,j < n$.

Solution by T. Amdeberhan, DeVry Institute, North Brunswick, NJ. Define the n-by-n matrix, $A_n(a,b)$, whose i,j–entry is $q^{(i+a-j-b)^2}$ for $0 \le i,j < n$. The proposed determinant is the special case $a = b = 0$. We claim that

$$\det A_n(a,b) = (-1)^{\binom{n}{2}} q^{n(a-b)^2} \prod_{k=1}^{n-1} (q^{2k}-1)^{n-k}. \tag{1}$$

To prove (1) we proceed by induction on n. The case $n = 1$ is trivial, so assume the formula holds for sizes less or equal to n. Then, according to Dodgson Rule for determinants,

$$\det A_{n+1}(a,b) = \frac{\det A_n(a,b)\det A_n(a+1,b+1) - \det A_n(a+1,b)\det A_n(a,b+1)}{\det A_{n-1}(a+1,b+1)}. \tag{2}$$

Applying the induction assumptions on the right-hand side of (2) and after simplifying the terms, it is now a simple matter to verify that (1) holds for $n+1$, too! □

References:

[**P**] P #10859, *American Mathematical Monthly,* **(108) #3,** March 2001.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX