SOLUTION TO PROBLEM #11309

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Proposed by Roman Witula and Damian Slota, Silesian University of Technology, Gliwice, Poland. Let γ and δ be real numbers satisfying $\sqrt{\gamma^2+\delta^2} < \frac{\pi}{2}$. Prove that $\cos(\gamma \sin x) > \sin(\delta \cos x)$ for all real x.

Solution by Tewodros Amdeberhan and Michael Joyce, Tulane Univesity, New Orleans, LA, USA. From the assumption, $|\gamma|, |\delta| < \frac{\pi}{2}$ and hence $|\gamma \sin x| < \frac{\pi}{2}$. So $\cos(\gamma \sin x) > 0$. The inequality is then trivially satisfied whenever $\delta \cos x \leq 0$; if $\delta, \gamma \geq 0$, then this observation and the periodicity of sine and cosine allow us to prove the inequality on the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. But we may use the transformations $\delta \to -\delta$, $x \to \pi - x$ and $\gamma \to -\gamma$, $x \to 2\pi - x$ to reduce to the case $\delta, \gamma \geq 0$. Furthermore, using $x \to -x$, we may in addition assume that $0 \leq x \leq \frac{\pi}{2}$.

Rewrite the claim as $\sin(\frac{\pi}{2} - \gamma \sin x) > \sin(\delta \cos x)$. Both arguments on the left and right-hand sides are in the interval between 0 and $\frac{\pi}{2}$. But, the sine function is 1-to-1 and increasing here. Thus the last assertion amounts to $\frac{\pi}{2} - \gamma \sin x > \delta \cos x$. On the other hand, using dot products and Cauchy-Schwartz:

$$\gamma \sin x + \delta \cos x = (\gamma, \delta) \cdot (\sin x, \cos x) \leq \sqrt{\gamma^2+\delta^2} < \frac{\pi}{2}.$$

The proof is complete. □

References:

[**P**] P #11309, *American Mathematical Monthly,* August-September 2007.