SOLUTION TO PROBLEM #11515 OF THE MONTHLY

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Proposed by Estelle L. Basor, American Institute of Mathematics, Palo Alto, CA, Steven N. Evans, University of California, Berkeley, CA, and Kent E. Morrison, California Polytechnic State University, San Luis Obispo, CA. Find a closed-form expression for

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\theta).$$

Solution. Since $\sin^4\beta = \frac{3}{8} - \frac{1}{2}\cos(2\beta) + \frac{1}{8}\cos(4\beta)$, the partial sum of the required series takes the form (including telescoping simplifications, regrouping and the identity $2\sin^2\beta = 1-\cos(2\beta)$)

$$\begin{aligned}\frac{3}{8}\sum_{n=1}^{N} 4^n - \frac{1}{2}\sum_{n=1}^{N} 4^n\cos(2^{1-n}\theta) + \frac{1}{8}\sum_{n=1}^{N} 4^n\cos(2^{2-n}\theta) &= \frac{1}{8}(4^{N+1}-4) + \frac{1}{2}\cos(2\theta) - \frac{4^N}{2}\cos(2^{1-N}\theta) \\ &= \frac{4^N}{2}[1-\cos(2^{1-N}\theta)] - \frac{1}{2}[1-\cos(2\theta)] \\ &= \theta^2\left(\frac{\sin(\theta/2^N)}{\theta/2^N}\right)^2 - \sin^2\theta.\end{aligned}$$

Since $\lim_{x\to 0}\frac{\sin x}{x} = 1$, the limit $N\to\infty$ implies that

$$\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\theta) = \theta^2 - \sin^2\theta. \qquad \square$$

Remark. A similar method shows that

$$\sum_{n=1}^{\infty} 4^{-n} \sin^4(2^{n}\theta) = \sin^2\theta\cos^2\theta.$$