Solution to Problem #11592
proposed by Mircea Ivan

Problem. *Find* $\lim_{n\to\infty}\left(-\log n+\sum_{k=1}^{n}\arctan(1/k)\right)$.

Solution by T. Amdeberhan and V.H. Moll, Tulane University, New Orleans, LA, USA. The proposed limit is reformulated as $\gamma+\lim_{n\to\infty}\left(\sum_{k=1}^{n}\arctan(1/k)-\sum_{k=1}^{n}1/k\right)$; where use is made of $\lim_{n\to\infty}\left(\sum_{k=1}\frac{1}{k}-\log n\right)=\gamma$ for the Euler gamma constant. The focus is then on the second limit $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\arctan\frac{1}{k}-\frac{1}{k}\right)$. Apply the expansion $\arctan x=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m-1}x^{2m-1}$ and some series manipulation to obtain

$$\begin{aligned}\sum_{k=1}^{n}\left(\arctan\frac{1}{k}-\frac{1}{k}\right)&=\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)k^{2m-1}}-\sum_{k=1}^{n}\frac{1}{k}\\&=\sum_{k=1}^{n}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-1)k^{2m-1}}\\&=\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-1)}+\sum_{k=2}^{n}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-1)k^{2m-1}}\\&=\frac{\pi}{4}-1+\sum_{k=2}^{n}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-1)k^{2m-1}}.\end{aligned}$$

Now, let $n\to\infty$ in the last double sum and invoke the Riemann zeta series so that

$$\sum_{k=2}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-1)k^{2m-1}}=\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-1)}\sum_{k=2}^{\infty}\frac{1}{k^{2m-1}}=\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{(2m-1)}\left(\zeta(2m-1)-1\right).$$

Define $F(x):=\sum_{m=2}^{\infty}x^{2m-2}\left(\zeta(2m-1)-1\right)$ and recall $\Psi(x)=\frac{d}{dx}\log\Gamma(x)$. It follows

$$F(x)=\sum_{k=2}^{\infty}\sum_{m=2}^{\infty}\frac{x^{2m-2}}{k^{2m-1}}=\sum_{k=2}^{\infty}\frac{-x^2}{k(x^2-k^2)}=1-\gamma-\frac{1}{2}\Psi(2+x)-\frac{1}{2}\Psi(2-x).$$

So, $\sum_{m=2}\frac{x^{2m-1}\,\infty}{\sum}{}_{k=2}\frac{1}{k^{2m-1}}=\int_0^x F(y)dy=x-\gamma x+\frac{1}{2}\log\left(\frac{\Gamma(2-x)}{\Gamma(2+x)}\right)$. For $x=\sqrt{-1}=i$, one can easily compute $\sum_{m=2}^{\infty}\frac{(-1)^m}{2m-1}\sum_{k=2}^{\infty}\frac{1}{k^{2m-1}}=-1+\gamma+\frac{i}{2}\log\left(\frac{\Gamma(2-i)}{\Gamma(2+i)}\right)$. Therefore

$$L:=\lim_{n\to\infty}\left(-\log n+\sum_{k=1}^{n}\arctan\frac{1}{k}\right)=\gamma+\frac{\pi}{4}-1-\left[-1+\gamma+\frac{i}{2}\log\left(\frac{\Gamma(2-i)}{\Gamma(2+i)}\right)\right].$$

Since $\Gamma(z+1)=z\Gamma(z),\overline{\Gamma(z)}=\Gamma(\overline{z}),\log(z)=\log(|z|)+i\cdot Arg(z)$, the required limit is

$$L=\frac{\pi}{4}-\frac{i}{2}\log\left[\frac{i\Gamma(-i)}{\Gamma(i)}\right]=\frac{\pi}{2}-\frac{i}{2}\log\left[\frac{\Gamma(-i)}{\Gamma(i)}\right]=\frac{\pi}{2}+\frac{1}{2}Arg\left[\frac{\Gamma(-i)}{\Gamma(i)}\right].$$

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