PROBLEM #11597
AMS MONTHLY

Problem #11597. *Let $f(x) = x/\log(1-x)$. For $0 < x < 1$, show that*

$$\sum_{k=1}^{\infty} \frac{x^k(1-x)^k}{k!} f^{(k)}(x) = -\frac{x}{2} f(x).$$

Proof. Recall the format of the Taylor series expansion of $f(y)$, at the number x, where the series converges.

$$f(y) = \sum_{k=0}^{\infty} \frac{(y-x)^k}{k!} f^{(k)}(x).$$

What should be y so that $y - x = x(1-x)$? This simply amounts to choosing $y = 2x - x^2$. The next question is, what is $f(y)$? Well, clearly

$$f(2x - x^2) = \frac{2x - x^2}{\log[1 - (2x - x^2)]} = \frac{2x - x^2}{\log[(1-x)^2]} = \frac{2x - x^2}{2\log(1-x)}.$$

On the other hand, splitting the summand for $k = 0$ results in

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(y-x)^k}{k!} f^{(k)}(x) &= \sum_{k=0}^{\infty} \frac{[x(1-x)]^k}{k!} f^{(k)}(x) \\ &= \frac{x}{\log(1-x)} + \sum_{k=1}^{\infty} \frac{x^k(1-x)^k}{k!} f^{(k)}(x). \end{aligned}$$

This says,

$$\frac{2x - x^2}{\log[(1-x)^2]} = \frac{x}{\log(1-x)} + \sum_{k=1}^{\infty} \frac{x^k(1-x)^k}{k!} f^{(k)}(x).$$

In other words,

$$\sum_{k=1}^{\infty} \frac{x^k(1-x)^k}{k!} f^{(k)}(x) = \frac{2x - x^2}{\log[(1-x)^2]} - \frac{x}{\log(1-x)} = \frac{-x^2}{2\log(1-x)} = -\frac{x}{2} f(x)$$

as required. □