SOLUTION TO PROBLEM #11810
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Tewodros Amdeberhan
tamdeber@tulane.edu

Problem #11810. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Napoca, Romania. Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$, and let ζ be the Riemann zeta function. Find

$$\sum_{n=1}^{\infty} H_n \left(\zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3} \right).$$

Proof. *Solution by Tewodros Amdeberhan, Tulane University, USA.* Start with the left-side:

$$\begin{aligned} \text{LHS} &= \sum_{n=1}^{\infty} H_n \sum_{k=n+1}^{\infty} \frac{1}{k^3} = \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{n=1}^{k-1} H_n = \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{n=1}^{k-1} \sum_{i=1}^{n} \frac{1}{i} = \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{i=1}^{k-1} \frac{1}{i} \sum_{n=i}^{k-1} 1 \\ &= \sum_{k=2}^{\infty} \frac{1}{k^3} \sum_{i=1}^{k-1} \frac{k-i}{i} = \sum_{k=2}^{\infty} \frac{H_{k-1}}{k^2} - \sum_{k=2}^{\infty} \frac{k-1}{k^3} = \zeta(3) - (\zeta(2) - \zeta(3)) = 2\zeta(3) - \zeta(2); \end{aligned}$$

where we recalled $\sum_k \frac{1}{k^2} H_{k-1} = \zeta(3)$ (due to Euler). The proof is complete. □

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