SOLUTION TO PROBLEM #11815
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Problem #11815. Proposed by George Apostolopoulos, Messolonghi, Greece. Let x, y and z be positive numbers such that $x+y+z=3$. Prove that

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1} \geq 3xyz.$$

Proof. *Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, USA.* The obvious reduction $\frac{a^4+a^2+1}{a^2+a+1}=a^2-a+1$ applied to the 3 variables (with $x+y+z=3$) implies

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}=x^2+y^2+z^2-(x+y+z)+3=x^2+y^2+z^2.$$

The claim now takes the equivalent form $x^2+y^2+z^2 \geq 3xyz$. Engaging the AGM inequality (twice):

$$x^2+y^2+z^2=(x^2+y^2+z^2)\left(\frac{x+y+z}{3}\right) \geq 3(xyz)^{\frac{2}{3}}(xyz)^{\frac{1}{3}}=3xyz. \qquad \square$$

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