SOLUTION TO PROBLEM #11819 OF THE AMERICAN MATHEMATICAL MONTHLY

Tewodros Amdeberhan & Victor H. Moll
tamdeber@tulane.edu vhm@tulane.edu

Problem #11819. Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA. Let f be a continuous non-negative function on $[0,1]$. Show that

$$\int_0^1 f^3(x)dx \geq 4\left(\int_0^1 x^2 f(x)\,dx\right)\left(\int_0^1 x f^2(x)\,dx\right).$$

Proof. *Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, USA.* The idea is to use Hölder's inequality $\int A(x)B(x)\,dx \leq \|A\|_{3/2}\|B\|_3$ twice:

$$\int_0^1 x^2 f(x)\,dx \leq \left(\int_0^1 (x^2)^{3/2}dx\right)^{2/3}\left(\int_0^1 f^3(x)\,dx\right)^{1/3},$$

$$\int_0^1 f^2(x)\,x\,dx \leq \left(\int_0^1 (f^2)^{3/2}dx\right)^{2/3}\left(\int_0^1 x^3\,dx\right)^{1/3}.$$

Multiplying out results in $\left(\int_0^1 x^2 f(x)\,dx\right)\left(\int_0^1 x f^2(x)\,dx\right) \leq \left(\int_0^1 x^3dx\right)\left(\int_0^2 f^3\,dx\right) = \frac{1}{4}\int_0^1 f^3\,dx$. The proof follows. $\square$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX