SOLUTION TO PROBLEM #11821
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Problem #11821. Proposed by Finbarr Holland and Clause Koester, University College Cork, Cork, Ireland. Let p be a positive integer. Prove that

$$\lim_{n\to\infty} \frac{1}{2^n n^p} \sum_{k=0}^{n} (n-2k)^{2p} \binom{n}{k} = \prod_{j=1}^{p} (2j-1).$$

Proof. *Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, USA.* Denote the quantity in the limit by $f_p(n)$. Rewriting $(n-2k)^{2p+2} = (n-2k)^{2p}(n^2-4k(n-k))$ and observing that $(n^2-4k(n-k))\binom{n}{k} = n^2\binom{n}{k} - 4n(n-1)\binom{n-2}{k-1}$ results in

$$f_{p+1}(n) = nf_p(n) - \frac{4(n-1)}{2^n n^p} \sum_{k=0}^{n} (n-2k)^{2p} \binom{n-2}{k-1} = nf_p(n) - \frac{n-1}{2^{n-2} n^p} \sum_{j=0}^{n-2} (n-2-2j)^{2p} \binom{n-2}{j}$$

which implies $f_{p+1}(n) - \left(\frac{n-2}{n}\right)^p f_p(n-2) = n\left[f_p(n) - \left(\frac{n-2}{n}\right)^p f_p(n-2)\right]$.
The desired identity is now verified by induction on p. The base case $p=1$: since $f_0(n) \equiv 1$ and from above recurrence, $f_1(n) - f_0(n-2) = n[f_0(n) - f_0(n-2)]$. So, $f_1(n) \equiv 1$. Assume the assertion holds for p. For n large enough, $\left(1-\frac{2}{n}\right)^p \sim 1 - \frac{2p}{n} + \binom{p}{2}\frac{4}{n^2}$. So, from the above recurrence:

$$\begin{aligned}\lim_{n\to\infty} f_{p+1}(n) - \lim_{n\to\infty} f_p(n-2) &= \lim_{n\to\infty} n\left[f_p(n) - f_p(n-2) + \frac{2p}{n} f_p(n-2) - \binom{p}{2}\frac{4}{n^2} f_p(n-2)\right] \\ &= \lim_{n\to\infty} n\left[f_p(n) - f_p(n-2)\right] + 2p \lim_{n\to\infty} f_p(n-2).\end{aligned}$$

If $S(j,i)$ denotes Stirling numbers of the second kind and $(x)_i := x(x-1)\cdots(x-i+1)$ then

$$\sum_{k=0}^{n} (n-2k)^{2p} \binom{n}{k} = \sum_{j=0}^{2p} (-2)^j \binom{2p}{j} n^{2p-j} \sum_{k=0}^{n} k^j \binom{n}{k} = 2^n \sum_{j=0}^{2p} (-1)^j \binom{2p}{j} n^{2p-j} \sum_{i=0}^{j} S(j,i)(n)_i 2^{j-i}$$

is a polynomial and $f_p(n) = \sum_{j=0}^{2p} (-1)^j \binom{2p}{j} \sum_{i=0}^{j} S(j,i) 2^{j-i} n^{2p-j} (n)_i$ a Laurent polynomial in n. In particular, combined with the induction hypothesis it holds $\lim_{n\to\infty} n[f_p(n) - f_p(n-2)] = 0$. So, $\lim_{n\to\infty} f_{p+1}(n) = (2p+1)\lim_{n\to\infty} f_p(n-2) = (2p+1)\prod_{j=1}^{p}(2j-1)$. The identity is proved. □

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