SOLUTION TO PROBLEM #11828
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Problem #11828. Proposed by Roberto Tauraso, Universita di Roma "Tor Vergata," Rome, Italy. Let n be a positive integer, and let z be a complex number that is not a kth root of unity for any k with $1 \leq k \leq n$. Let S be the set of all lists $(a_1, \dots, a_n)$ of n nonnegative integers such that $\sum_{k=1}^{n} ka_k = n$. Prove that

$$\sum_{a\in S}\prod_{k=1}^{n}\frac{1}{a_k!k^{a_k}(1-z^k)^{a_k}} = \prod_{k=1}^{n}\frac{1}{1-z^k}.$$

Proof. Standard exponential generating function techniques (see e.g. [1, Eqn. (5.30)]) show a result due to Touchard:

$$\sum_{n=0}^{\infty}\left(\frac{1}{n!}\sum_{\pi\in S_n}u_1^{c_1}u_2^{c_2}\cdots u_n^{c_n}\right)t^n = e^{u_1\frac{t^1}{1}+u_2\frac{t^2}{2}+u_3\frac{t^3}{3}+\cdots}; \tag{1}$$

where $c_i = c_i(\pi)$ denotes the number cycles of length i in a permutation π. If $\pi \in S_n$ then its cycle type $(a_1, \dots, a_n) \vdash n$ is a partition. It's also known that there are $\prod_{k=1}^{n}\frac{k}{a_k!k^{a_k}}$ such permutations, and hence equation (1) takes the desired form (replacing $u_k = \frac{1}{1-z^k}$)

$$\sum_{n=0}^{\infty}\left(\sum_{a\vdash n}\prod_{k=1}^{n}\frac{1}{a_k!k^{a_k}(1-z^k)^{a_k}}\right)t^n = e^{\sum_{n=1}^{\infty}\frac{t^n}{n(1-z^n)}}.$$

On the other hand, $\sum_{n=1}^{\infty}\frac{t^n}{n(1-z^n)} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{t^nz^{nk}}{n} = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(tz^k)^n}{n} = -\sum_{k=0}^{\infty}\log(1-tz^k)$ so that

$$e^{\sum_{n=1}^{\infty}\frac{t^n}{n(1-z^n)}} = e^{-\sum_{k=0}^{\infty}\log(1-tz^k)} = \prod_{k=0}^{\infty}\frac{1}{1-tz^k}. \tag{3}$$

Now, the coefficient of t^n in (3) is the generating function for partitions of N with largest part at most n, which is $\prod_{k=1}^{n}\frac{1}{1-z^k}$. The equality is clearly valid for $|z| < 1$, but as rational meromorphic functions they must agree over $\mathbb{C}$ beside the poles. The proof follows. □

REFERENCES

[1] R P Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge **62** (1999).

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