SOLUTION TO PROBLEM #11832 OF THE AMERICAN MATHEMATICAL MONTHLY

Tewodros Amdeberhan & Victor H. Moll
tamdeber@tulane.edu vhm@tulane.edu

Problem #11832. Proposed by Donald Knuth, Stanford University, Stanford, CA. Let $C(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$ (thus $C(z)$ is the generating function of the Catalan numbers). Prove that

$$[\log C(z)]^2 = \sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n}.$$

Here, $H_k = \sum_{j=1}^{k} \frac{1}{j}$; that is, H_k is the kth *harmonic number.*

Proof. *Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, USA.* Let $g(z) = \log C(z)$. Recall that $C(z) = \frac{1-\sqrt{1-4z}}{2z}$. If $g(z) := \log C(z)$ then $g'(z) = \frac{1}{2z}\left[\frac{1}{\sqrt{1-4z}} - 1\right] = \frac{1}{2}\sum_{n=1}^{\infty}\binom{2n}{n} z^{n-1}$ and $g(0) = 0$, hence $g(z) = \sum_{n=1}^{\infty}\binom{2n-1}{n}\frac{z^n}{n}$. Cauchy's product rule gives

$$[g(z)]^2 = [\log C(z)]^2 = \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n-1}\binom{2k-1}{k}\binom{2n-2k-1}{n-k}\frac{1}{k(n-k)}\right] z^n.$$

Denoting $F(n,k) := \frac{1}{2}\binom{2k-1}{k}\binom{2n-2k}{n-k}\binom{2n}{n}^{-1}\frac{n}{k(n-k)}$, it suffices to verify $\sum_{k=1}^{n-1} F(n,k) = H_{2n-1} - H_n$. To this end, Zeilberger's algorithm generates $G(n,k) := -\frac{F(n,k)(2n-2k+1)(n-k)(3n-2k+3)k^2}{n(n+1)(2n+1)(n-k+1)^2}$ to satisfy

$$\frac{F(n+1,k)}{F(n,k)} - 1 = -\frac{(3n^2+4n-3kn+1-2k)k}{n(n+1-k)^2(2n+1)} = \frac{G(n,k+1)-G(n,k)}{F(n,k)}.$$

Therefore, $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$. Summing over the integers $k=1$ through $k=n-1$ and telescoping on the right-hand side, yield

$$\sum_{k=1}^{n-1} F(n+1,k) - \sum_{k=1}^{n-1} F(n,k) = G(n,n) - G(n,1) = \frac{3n+1}{4n(n+1)(2n+1)} - \frac{n(n+3)}{4(n+1)(2n+1)}.$$

On the other hand, $(H_{2n+1} - H_{n+1}) - (H_{2n-1} - H_n) - F(n+1,n) = \frac{3n+1}{4n(n+1)(2n+1)} - \frac{n(n+3)}{4(n+1)(2n+1)}$ which means $\sum_{k=1}^{n} F(n+1,k) - \sum_{k=1}^{n-1} F(n,k) = (H_{2n+1} - H_{n+1}) - (H_{2n-1} - H_n)$. Upon checking initial condition (say $n=1$), the proof follows. □

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