SOLUTION TO PROBLEM #11834
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Problem #11834. Proposed by Arkady Alt, San Jose, CA. For nonnegative real numbers u, v, w, let $\Delta(u,v,w) = 2(uv+vw+wu) - (u^2+v^2+w^2)$. Say that two lists (a,b,c) and (x,y,z) *agree in order* if $(a-b)(x-y) \geq 0, (b-c)(y-z) \geq 0$, and $(c-a)(z-x) \geq 0$. Prove that if (x,y,z) and (a,b,c) agree in order, then $\Delta(a,b,c)\Delta(x,y,z) \geq 3\Delta(ax,by,cz)$.

Proof. *Solution by Tewodros Amdeberhan, Tulane University, USA.* Note: the relation *agree in order* and the inequality are invariant under joint cyclic shifts of (a,b,c) and (x,y,z). In a similar vain (upto cyclic reordering), there are exactly two possibilities:

(i) $a \geq b \geq c \geq 0$ and $x \geq y \geq z \geq 0$ or (ii) $0 \leq a \leq b \leq c$ and $0 \leq x \leq y \leq z$.

We only consider (i), the second case is similar. Rewrite $\Delta(a,b,c)\Delta(x,y,z) - 3\Delta(ax,by,cz)$ as:

$$\begin{aligned}\Delta(a,b,c)\Delta(x,y,z) - 3\Delta(ax,by,cz) &= (a-b)^2[4(x-y)^2+3(y-z)^2] + 3(b-c)^2(x-y)^2 \\ &+ 2(a-b)^2[(x-y)(3y-z)+z(y-z)] \\ &+ 2(a-b)[(x-y)^2(3b-c)+(x-y)(3by-cz)+cz(y-z)] \\ &+ 2c(b-c)[(x-y)^2+z(x-z)+z(y-z))] \geq 0,\end{aligned}$$

because each summand is nonnegative due to (i). The proof is complete. □

Remark. The choice $(a,b,c) = (x,y,z)$ recovers a special case of *Schur's inequality*

$$\Delta(a,b,c)^2 - 3\Delta(a^2,b^2,c^2) = 4[a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b)] \geq 0.$$

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