SOLUTION TO PROBLEM #11837 OF THE AMERICAN MATHEMATICAL MONTHLY

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Problem #11837. Proposed by Iosif Pinelis, Michigan Technological University, Houghton, MI. Let $a_0 = 1$, and for $n \geq 0$ let $a_{n+1} = a_n + e^{-a_n}$. Let $b_n = a_n - \log n$. For $n \geq 0$, show that $0 < b_{n+1} < b_n$ and also show that $\lim_{n\to\infty} b_n = 0$.

Proof. *Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, USA.* Obviously $a_n \geq 1$ is strictly increasing (think of $f(x) = x + e^{-x}$) and its limit can not be finite (else, the recurrence leads to a contradiction). So, $\lim_{n\to\infty} a_n = \infty$. Define the sequence $c_n = e^{a_n}$ whose limit is also ∞. Now, consider the difference

$$c_{n+1} - c_n = e^{a_n + e^{-a_n}} - e^{a_n} = e^{a_n}\left(e^{e^{-a_n}} - 1\right) = c_n\left(e^{\frac{1}{c_n}} - 1\right) = \frac{e^{\frac{1}{c_n}} - 1}{\frac{1}{c_n}}.$$

It is now evident that the sequence $d_{n+1} := c_{n+1} - c_n$ satisfies: (1) it is decreasing (since $\frac{e^x-1}{x}$ is increasing and $\frac{1}{c_n}$ is decreasing); (2) its limit is 1 (think of $\lim_{x\to 0}\frac{e^x-1}{x} = 1$). Because of (2) and by Cesaro's Mean: $\lim_{n\to\infty}\frac{c_n}{n} = 1$. That means,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n - \log n) = \lim_{n\to\infty}\log\left(\frac{c_n}{n}\right) = 0.$$

From property (1), the average $\frac{c_n}{n}$ of d_n is strictly decreasing hence so is $b_n = \log\left(\frac{c_n}{n}\right)$. Next, proceed with a left endpoint Riemann sum approximation over the partitions $[a_k, a_{k+1}]$ (for $0 \leq k \leq n-1$) so that

$$e^{a_n} - e = \int_1^{a_n} e^x\,dx > \sum_{k=0}^{n-1}(a_{k+1} - a_k)e^{a_k} = \sum_{k=0}^{n-1} 1 = n$$

which implies that $a_n > \log(n+e) > \log n$ or $b_n = a_n - \log n > 0$. The proof follows. $\square$

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