SOLUTION TO PROBLEM #11843
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Problem #11843. Proposed by Mihali Bencze, Bucharest, Romania. Let n and k be positive integers, and let $x_j \geq 1$ for $1 \leq j \leq n$. Let $y = \prod_{i=1}^n x_i$. Show that

$$\sum_{i=1}^n \frac{1}{1+x_i} \geq \sum_{j=1}^n \frac{1}{1+(x_j^{k-1}y)^{1/(n+k-1)}}.$$

Proof. *Solution by Tewodros Amdeberhan, Tulane University, USA.* Recall Jensen's inequality for convex functions: $f\left(\frac{\sum_{i=1}^n \lambda_i a_i}{\sum_{j=1}^n \lambda_j}\right) \geq \sum_{i=1}^n \frac{\lambda_i f(a_i)}{\sum_{j=1}^n \lambda_j}$ for positive numbers λ_j. The function $f(x) = \frac{1}{1+e^x}$ is convex since $f''(x) = \frac{e^x(e^x-1)}{(1+e^x)^3} > 0$ whenever $x \in (0,\infty)$. Denote $\lambda = (\lambda_1, \dots, \lambda_n)$. Apply Jensen's inequality to f repeatedly n times: for $\lambda = (k,1,1,\dots,1)$; for $\lambda = (1,k,1,\dots,1)$; and so on, until $\lambda = (1,1,1,\dots,k)$. In all cases, $\sum_i \lambda_i = n+k-1$. These lead to a system of n inequalities; for wit, the first and the last inequalities appear respectively as

$$\frac{1}{n+k-1}\left[\frac{k}{1+e^{a_1}} + \frac{1}{1+e^{a_2}} + \cdots + \frac{1}{1+e^{a_n}}\right] \geq \frac{1}{1+e^{\frac{ka_1+a_2+\cdots+a_n}{n+k-1}}},$$
$$\frac{1}{n+k-1}\left[\frac{1}{1+e^{a_1}} + \frac{1}{1+e^{a_2}} + \cdots + \frac{k}{1+e^{a_n}}\right] \geq \frac{1}{1+e^{\frac{a_1+a_2+\cdots+ka_n}{n+k-1}}}.$$

Adding all n inequalities produces

$$\sum_{i=1}^n \frac{1}{1+e^{a_i}} \geq \sum_{j=1}^n \frac{1}{1+\left(e^{(k-1)a_j}e^{a_1+\cdots+a_n}\right)^{1/(n+k-1)}}.$$

Now replace $x_i = e^{a_i} \geq 1$. The proof is complete. □

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