SOLUTION TO PROBLEM #11844 OF THE AMERICAN MATHEMATICAL MONTHLY

Tewodros Amdeberhan & Shalosh B. Ekhad
tamdeber@tulane.edu

Problem #11844. Proposed by H. Ohtsuka (Japan) and R. Tauraso (Italy). For nonnegative integers m and n, prove

$$\sum_{k=0}^{n}(m-2k)\binom{m}{k}^3 = (m-n)\binom{m}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1}.$$

Proof. *Solution by Tewodros Amdeberhan, Tulane University, and Shalosh B. Ekhad, USA.* Let $F_1(m,k)=(m-2k)\binom{m}{k}^3$ and $F_2(m,j)=(m-n)\binom{m}{n}\binom{j}{n}\binom{j}{m-n-1}$. Notice that the assertion amounts to $f_1(m):=\sum_{k=0}^{n}F_1(m,k)=\sum_{j=0}^{m-1}F_2(m,j):=f_2(m)$. Zeilberger's algorithm generates the WZ-mates $G_1(m,k)=(2m-k+2)\binom{m}{k-1}^3$ and $G_2(m,j)=(n+1)\binom{m}{n}\binom{j}{j+1}\binom{j}{m-n}$ so that

$$F_1(m{+}1,k)+F_1(m,k)=G_1(m,k{+}1)-G_1(m,k) \qquad F_2(m{+}1,j)+F_2(m,j)=G_2(m,j{+}1)-G_2(m,j).$$

Summing over k and j, respectively and observing telescoping properties with G_1, G_2, we find that

$$\sum_{k=0}^{n}F_1(m+1,k)+\sum_{k=0}^{n}F_1(m,k)=\sum_{k=0}^{n}G_1(m,k+1)-\sum_{k=0}^{n-1}G_1(m,k+1)=(2m-n+1)\binom{m}{n}^3,$$

$$\sum_{j=0}^{m-1}F_2(m+1,j)+\sum_{j=0}^{m-1}F_1(m,j)=\sum_{j=0}^{m-1}G_2(m,j+1)-\sum_{j=0}^{m-1}G_2(m,j)=m\binom{m-1}{n}\binom{m}{n}^2.$$

The first of these equations offers the recurrence $f_1(m{+}1)+f_1(m)=(2m{-}n{+}1)\binom{m}{n}^2$. After adding the term $F_2(m+1,m)$ to both sides of the second equation, we are lead to $f_2(m+1)+f_2(m)=m\binom{m-1}{n}\binom{m}{n}^2+(m+1-n)\binom{m+1}{n}\binom{m}{n}^2=(2m-n+1)\binom{m}{n}^3$. The two functions f_1 and f_2 satisfy the same linear recurrence. Note: when the *suppressed* variable $n\geq m$, $f_2(m)=0$ trivially, while $f_1(m)=\sum_{k=0}^{m}(m-k)\binom{m}{m-k}^3-\sum_{k=0}^{m}k\binom{m}{k}^3=0$. Checking $f_1(0)=f_2(0)=0$ match, the identity $f_1(m)=f_2(m)$ follows. $\square$