SOLUTION TO PROBLEM #11847
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Tewodros Amdeberhan
tamdeber@tulane.edu

Problem #11847. *Proposed by Mihaly Bencze, Brasov, Romania.* Prove that for $n \geq 1$,

$$\frac{n(n+1)(n+2)}{3} < \sum_{k=1}^{n} \frac{1}{\log^2(1+1/k)} < \frac{n}{4} + \frac{n(n+1)(n+2)}{3}.$$

Proof. *Solution by Tewodros Amdeberhan, Tulane University, USA.* Since $\sum_{k=1}^{n} \frac{1}{4} = \frac{n}{4}$ and $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$, it suffices to show the *point-wise* (term-by-term) estimates $k(k+1) < \frac{1}{\log^2(1+1/k)} < \frac{1}{4} + k(k+1) = (k+1/2)^2$, equivalently $\frac{1/k}{\sqrt{1+1/k}} > \log(1+1/k) > \frac{1/k}{1+k/2}$. Define $f(x) = \frac{x}{\sqrt{1+x}} - \log(1+x)$ and $g(x) = \log(1+x) - \frac{x}{1+x/2}$. These are strictly increaasing, since

$$f'(x) = \frac{\sqrt{x^2+4x+4} - \sqrt{4x+4}}{2(1+x)^{3/2}} > 0 \qquad \text{and} \qquad g'(x) = \frac{x^2}{(1+x)(2+x)^2} > 0;$$

provided $x > 0$. Therefore, $f(x) > f(0) = 0$ and $g(x) > g(0) = 0$. Plugging in $x = \frac{1}{k}$ (for $k \geq 1$) implies $f(\frac{1}{k}) > 0$ and $g(\frac{1}{k}) > 0$. The proof follows. □