SOLUTION TO PROBLEM #11850
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Problem #11850. *Proposed by Zafar Ahmed, Bhabha Atomic Research Center, Mumbai, India.* Let A_n be the function given by

$$A_n(x)=\sqrt{\frac{2}{\pi}}\,\frac{1}{n!}\,(1+x^2)^{n/2}\frac{d^n}{dx^n}\left(\frac{1}{1+x^2}\right).$$

Prove that for nonnegative integers m and n, $\int_{-\infty}^{\infty}A_m(x)A_n(x)dx=\delta(m,n)$, where $\delta(m,n)=1$ if $m=n$, and otherwise $\delta(m,n)=0$.

Proof. *Solution by Tewodros Amdeberhan, Tulane University, and Hade Kilete-Seleste, USA.* Let $f(x)=\frac{1}{1+x^2}$ and denote $D=\frac{d}{dx}$. The following can be proved by induction:

$$D^nf(x)=-f(x)\left[2nxD^{n-1}f(x)+n(n-1)D^{n-2}f(x)\right],$$

$$D^nf(x)=n!\sum_{k=0}^{\lfloor n/2\rfloor}(-2)^{n-k}2^{-k}\binom{n-k}{k}x^{n-2k}f^{n+1-k},$$

$$\frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos t)^2(\sin t)^{2n}dt=\frac{C(n)}{2^{2n}}\qquad\text{and}\qquad\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos t)^2(\sin t)^{2n+1}dt=0;$$

where $C(n)=\frac{1}{n+1}\binom{2n}{n}$ are the Catalan numbers. Now, the substitution $x=\tan t$ leads to

$$\begin{aligned}\int_{\mathbb{R}}A_m(x)A_n(x)dx&=\frac{2}{\pi}\sum_{j,k\geq 0}(-2)^{m+n-j-k}2^{-j-k}\binom{m-j}{j}\binom{n-k}{k}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos t)^2(\sin t)^{m+n-2k-2j}dt\\&=\sum_{j,k\geq 0}(-1)^{j+k}\binom{m-j}{j}\binom{n-k}{k}C\left(\frac{m+n}{2}-j-k\right);\end{aligned}$$

where m and n are forcibly of the *same parity,* else the integral is 0. Zeilberger's algorithm generates

$$\sum_{j,k\geq 0}=\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\binom{m-j}{j}\frac{n+1}{\frac{m+n}{2}-j+1}\binom{m-2j}{\frac{m+n}{2}-j}.$$

Observe that $\binom{m-2j}{\frac{m+n}{2}-j}=0$ unless $j=0$ and $m=n$, in which case $\int_{\mathbb{R}}A_m(x)A_m(x)dx=\sum_{j,k\geq 0}=1$. The proof is complete. □

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