SOLUTION TO PROBLEM #11853
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Problem #11853. Proposed by H. Ohtsuka, Japan. Find

$$\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n)}.$$

Solution by Tewodros Amdeberhan, Tulane University; Armin Straub, University of South Alabama. We actually show that $\sum_{n=1}^{\infty} \frac{1}{\sinh(2^n x)} = \coth(x) - 1$, for $x > 0$. The familiar addition identity $\sinh(x) = \sinh(2x - x) = \sinh(2x)\cosh x - \cosh(2x)\sinh(x)$ divided through by $\sinh(2x)\sinh(x)$ implies $\frac{1}{\sinh(2x)} = \coth(x) - \coth(2x)$. A repeated application of the latter identity and telescoping lead to

$$\sum_{n=1}^{N} \frac{1}{\sinh(2^n x)} = \sum_{n=1}^{N} [\coth(2^{n-1}x) - \coth(2^n x)] = \coth(x) - \coth(2^N x).$$

Observing $\coth(2^N x) = \frac{e^{2^N x} + e^{-2^N x}}{e^{2^N x} - e^{-2^N x}} \to 1$, as $N \to \infty$, completes the argument. $\square$

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