SOLUTION TO PROBLEM #11855 OF THE AMERICAN MATHEMATICAL MONTHLY

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Problem #11855. Proposed by Cezar Lupu, USA. For a continuous and non-negative function f on $[0,1]$, let $\mu_n = \int_0^1 x^n f(x)\,dx$. Show that $\mu_{n+1}\mu_0 \geq \mu_n\mu_1$ for $n \in \mathbb{N}$.

Solution by Tewodros Amdeberhan, Tulane University; Armin Straub, University of South Alabama. Rewrite the inequality as $\int_0^1\int_0^1 x^{n+1}F(x,y)\,dxdy - \int_0^1\int_0^1 x^n y\,F(x,y)\,dxdy \geq 0$, with $F(x,y) = f(x)f(y)$. We prove the inequality for the Riemann sum and pass on to the limit. Divide the unit square into an m^2 grid, then select $(\frac{j}{m},\frac{k}{m})$ for $1 \leq j,k \leq m$ and denote $a_{jk} = F(\frac{j}{m},\frac{k}{m})$. Observe that $a_{jk} \geq 0$ and the symmetry $a_{jk} = a_{kj}$. For the Riemann sum, the inequality take the form

$$\frac{1}{m^2}\sum_{j,k} a_{jk}\left(\frac{j}{m}\right)^{n+1} - \frac{1}{m^2}\sum_{j,k} a_{jk}\left(\frac{j}{m}\right)^{n}\left(\frac{k}{m}\right) \geq 0.$$

Multiplying through by m^{n+3}, we rearrange (e.g. some renaming in between) the left-hand side

$$\begin{aligned}\sum_{j,k} j^n(j-k)a_{jk} &= \sum_{j>k} j^n(j-k)a_{jk} + \sum_{j<k} j^n(j-k)a_{jk} \\ &= \sum_{j>k} j^n(j-k)a_{jk} + \sum_{k<j} k^n(k-j)a_{kj} \\ &= \sum_{j>k} j^n(j-k)a_{jk} + \sum_{k<j} k^n(k-j)a_{jk} \\ &= \sum_{j>k} (j-k)(j^n-k^n)a_{jk},\end{aligned}$$

which is of course non-negative. The inequality holds for all Riemann sums corresponding to m, hence it is valid the double integral as well. $\square$

Remark. The same reasoning (verbatim) captures $\mu_{n+r+s}\mu_s \geq \mu_{n+s}\mu_{r+s}$ for $n,r,s \in \mathbb{R}^+$.

Remark. Your generalization abound, let's view this as follows. For a non-negative (finite) measure ν on $S \subset \mathbb{R}$. Turn this to a probability measure $\frac{\nu(x)}{|\nu|}$ where $|\nu| = \int_S \nu(x)$. Now, consider the family of measure $\nu_n(x) = x^n f(x)dx$ where $f \geq 0$. Then, one interpretation of the inequality $\mu_{n+r}\nu_0 \geq \mu_n\mu_r$ runs as follows. The r^{th}-moments of the measures $\frac{\nu_n}{|\nu_n|}$ are no less than the r^{th}-moments of that of $\frac{\nu_0}{|\nu_0|}$; i.e.

$$\int_S x^r\frac{\nu_n(x)}{|\nu_n|} \geq \int_S x^r\frac{\nu_0(x)}{|\nu_0|}.$$

Consequently, we can say that $\frac{\nu_0(x)}{|\nu_0|}$ is *absolutely continuous* with respect to $\frac{\nu_n(x)}{|\nu_n|}$. This is already evident from the Radon-Nikodym derivative $\frac{d\nu_0}{d\nu_n} = x^n$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX