SOLUTION TO PROBLEM #11867
OF THE AMERICAN MATHEMATICAL MONTHLY

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Problem #11867. *Proposed by George Apostolopoulos, Messolonghi, Greece.* For real numbers a, b, c, let

$$f(a,b,c) = \left(\frac{a^2}{a^2-ab+b^2}\right)^{1/4}.$$

Prove that $f(a,b,c) + f(b,c,a) + f(c,a,b) \leq 3$.

Proof. *Solution by Tewodros Amdeberhan, Tulane University, USA.* From $(a-b)^2 \geq 0$, we get $a^2-ab+b^2 \geq ab$ and then $3(a^3+b^3) \geq 3ab(a+b)$. So, $4(a^3+b^3) \geq a^3+b^3+3ab(a+b) = (a+b)^3$. Hence $\frac{1}{a^3+b^3} \leq \frac{4}{(a+b)^3}$. Consequently, for the problem at hand, we obtain

$$\sum_{cyc}\left(\frac{a^2}{a^2-ab+b^2}\right)^{1/4} = \sum_{cyc}\left(\frac{a^2(a+b)}{a^3+b^3}\right)^{1/4} \leq \sum_{cyc}\left(\frac{4a^2(a+b)}{(a+b)^3}\right)^{1/4} = \sum_{cyc}\sqrt{\frac{2a}{a+b}}.$$

At this point, Jensen's inequality applied to the concave function $f(x) = \sqrt{x}$ effectively yields

$$\begin{aligned}\sum_{cyc}\sqrt{\frac{2a}{a+b}} &= \sum_{cyc}\frac{a+c}{2(a+b+c)}\sqrt{\frac{8a(a+b+c)^2}{(a+b)(a+c)^2}} \\ &\leq \sqrt{\sum_{cyc}\frac{a+c}{2(a+b+c)}\frac{8a(a+b+c)^2}{(a+b)(a+c)^2}} = \sqrt{\sum_{cyc}\frac{4a(a+b+c)}{(a+b)(a+c)}}.\end{aligned}$$

It remains to show $\sum_{cyc}\frac{4a(a+b+c)}{(a+b)(a+c)} \leq 9$. Clearing denominators, noting $\sum_{cyc} a(b+c) = 2\sum_{cyc} ab$, and expanding the resulting expressions, this last claim amounts to

$$8\sum_{cyc} ab \cdot \sum_{cyc} a \leq 9\prod_{cyc}(a+b) \qquad \Leftrightarrow \qquad \sum_{cyc} c(a-b)^2 \geq 0.$$

The above argument *implicitly* assumes a, b, c to be non-negative. Such is no loss of generality because for negative numbers, $f(a,b,c)$ simply becomes smaller. The proof is complete. □

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