SOLUTION TO PROBLEM #11872
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Problem #11872. *Proposed by Phu Cuong Le Van, College of Education, Hue, Vietnam.* Let f be a continuous function from $[0,1]$ to $\mathbb{R}$ such that $\int_0^1 f(x)dx = 0$. Prove that for each positive integers there exists $c \in (0,1)$ such that

$$n\int_0^c f(x)dx = c^{n+1}f(c).$$

Proof. *Solution by Tewodros Amdeberhan, Tulane University, USA.* First, we show that if $g : [0,b] \to \mathbb{R}$ is continuous and $\int_0^b g(x)dx = 0$ then $\int_0^c xg(x)dx = 0$ for some $c \in (0,b)$. Suppose not. Since $A(t) := \int_0^t xg(x)dx$ is continuous, either $A(t) > 0$ or $A(t) < 0$ for all $t \in (0,b)$. WLOG assume $A(t) > 0$. Denote $B(t) = \int_0^t g(x)dx$. Writing $xg(x) = (xB(x))' - B(x) = B(x) + xg(x) - B(x)$, we obtain $A(t) = tB(t) - \int_0^t B(x)dx > 0$ for all $t \in (0,b)$. By a limiting process, $bB(b) - \int_0^b B(x)dx \geq 0$ or $\int_0^b B(x)dx \leq 0$ (since $B(b) = 0$). Define $h : [0,b] \to \mathbb{R}$ continuous, and differentiable in $(0,b)$, by

$$h(t) = \begin{cases} \frac{1}{t}\int_0^t B(x)dx & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then, $h'(t) = \frac{1}{t^2}(tB(t) - \int_0^t B(x)dx) > 0$ throughout $(0,b)$ (see above). By the Mean Value Theorem, $h(b) - h(0) = h'(a)(b-0) > 0$ for some $a \in (0,b)$. It follows that $h(b) = \frac{1}{b}\int_0^b B(x)dx > 0$, which is a contradiction. Therefore, there exists $c \in (0,b)$ such that $A(c) = \int_0^c xg(x)dx = 0$.

Apply this result to $g(x) = f(x)$ (with $b = 1$) to get $\int_0^{c_1} xf(x)dx = 0$, then to $g(x) = xf(x)$ (with $b = c_1$) to obtain $\int_0^{c_2} x^2 f(x)dx = 0$, and so on. Hence $\int_0^{c_n} x^n f(x)dx = 0$ for some $c_n \in (0,1)$. Let

$$E(t) = \begin{cases} \frac{1}{t^n}\int_0^t x^n f(x)dx & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

The function E is continuous on $[0,c_n]$, differentiable in $(0,c_n)$ and $E(0) = E(c_n) = 0$. By Rolle's Theorem, there exists $\eta_n \in (0,c_n)$ such that $0 = E'(\eta_n) = \frac{-n}{\eta_n^{n+1}}\int_0^{\eta_n} x^n f(x)dx + f(\eta_n)$. That means,

$$n\int_0^{\eta_n} x^n f(x)dx = \eta_n^{n+1} f(\eta_n). \qquad \square.$$