SOLUTION TO PROBLEM #11873
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Problem #11873. *Proposed by E. J. Ionascu, USA.* Show that for $n \in \mathbb{N}$ with $n \geq 2$,

$$\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\cot\frac{(2k-1)\pi}{2n}=\sum_{k=1}^{n-1}\csc\frac{k\pi}{n}.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Start with the (obvious) facts: $\sum_{k=1}^{n}x^k=\frac{x^{n+1}-x}{x-1}$ and $\sum_{k=1}^{n}kx^k=x\frac{d}{dx}\frac{x^{n+1}-x}{x-1}$. Letting $x=e^{\frac{\pi ir}{n}}$ and since $x^{2n}=1$, we obtain $\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)x^{2k-1}=\frac{2x}{1-x^2}=\frac{-2}{x-\frac{1}{x}}=i\cdot\csc\frac{\pi r}{n}$. If $y=e^{\frac{\pi i(2k-1)}{n}}$ then $y^n=-1$ and hence

$$\begin{aligned} i\cdot\sum_{r=1}^{n-1}\csc\frac{\pi r}{n}&=\sum_{r=1}^{n-1}\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)x^{2k-1}=\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\sum_{r=1}^{n-1}y^r\\ &=-\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\frac{y+1}{y-1}=i\sum_{k=1}^{n}\left(1-\frac{2k-1}{n}\right)\cot\frac{(2k-1)\pi}{2n}. \end{aligned}$$

This completes the argument. □

DFT. Let $f(n)$ be a periodic sequence, of period k. Denote $\zeta=e^{\frac{2\pi i}{k}}$. Then, the *discrete Fourier transform* and its inverse are defined, respectively, by

$$\hat{f}(n)=\sum_{j=0}^{k-1}f(j)\zeta^{jn}\qquad\text{and}\qquad f(n)=\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\zeta^{-jn}.$$

Plancherel's formula. *If f and g have period k, then*

$$\sum_{j=0}^{k-1}f(j)g(j)=\frac{1}{k}\sum_{j=0}^{k-1}\hat{f}(j)\hat{g}(j).$$

QUESTION. Is it possible to find two appropriate functions f and g so that the identity in the above Monthly problem can be recovered from Plancherel's formula? It certainly seems so, but I don't know how.

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