SOLUTION TO PROBLEM #11874
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Problem #11874. *Proposed by C. I. Valean, Romania.* Evaluate the limits below,

$$\lim_{n\to\infty}\sum_{k=2}^{n-1}\frac{\zeta(k)}{\Gamma(n-k)}, \qquad \lim_{n\to\infty}\sum_{k=1}^{n-2}(-1)^{k-1}\frac{\zeta(n-k)}{\Gamma(k)}$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Consider $a_n := \sum_{k=0}^{n-2}\frac{\zeta(n-k)}{k!}$ and $b_n := \sum_{k=0}^{n-2}\frac{1}{k!}$. Now, let's compute the sum

$$\sum_{n\geq 2}(a_n-b_n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\zeta(n-k+2)-1}{k!}=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=k}^{\infty}(\zeta(n-k+2)-1)=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=2}^{\infty}(\zeta(n)-1)$$
$$=e\cdot\sum_{n=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{j^n}=e\sum_{j=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{j^n}=e\sum_{j=2}^{\infty}\frac{1}{j(j-1)}=e.$$

Since the series is convergent, $\lim_{n\to\infty}(a_n-b_n)=0$ and hence $\lim_{n\to\infty}a_n=\lim_{n\to\infty}(b_n+a_n-b_n)=\lim_{n\to\infty}b_n+\lim_{n\to\infty}(a_n-b_n)=\sum_{k=0}^{\infty}\frac{1}{k!}=e$.

The second limit is worked out similarly. Let $c_n := \sum_{k=0}^{n-2}\frac{(-1)^k\zeta(n-k)}{k!}$ and $d_n := \sum_{k=0}^{n-2}\frac{(-1)^k}{k!}$. So,

$$\sum_{n\geq 2}(c_n-d_n)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{n=k}^{\infty}(\zeta(n-k+2)-1)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{n=2}^{\infty}(\zeta(n)-1)=\frac{1}{e}.$$

Since the series is convergent, $\lim_{n\to\infty}(c_n-d_n)=0$ and hence $\lim_{n\to\infty}c_n=\lim_{n\to\infty}(d_n+c_n-d_n)=\lim_{n\to\infty}d_n+\lim_{n\to\infty}(c_n-d_n)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}=\frac{1}{e}$. $\square$

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