SOLUTION TO PROBLEM #11875
OF THE AMERICAN MATHEMATICAL MONTHLY

Tewodros Amdeberhan
tamdeber@tulane.edu

Problem #11875. *Proposed by D. M. Batinetu-Girugiu and N. Stanciu, Romania.* Let $f_n = (1+1/n)^n((2n-1)!!L_n)^{1/n}$. Find $\lim_{n\to\infty}(f_{n+1}-f_n)$ where L_n denotes the nth Lucas number (given by $L_0=2, L_1=1$, and by $L_n = L_{n-1}+L_{n-2}$ for $n\geq 2$).

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Denote $a_n = (1+1/n)^n$ and $b_n = ((2n-1)!!L_n)^{1/n}$ so that $f_{n+1}-f_n = a_{n+1}(b_{n+1}-b_n) + \frac{b_n}{n}(a_{n+1}-a_n)n$. We work out individual limits: (**1**) $\lim a_{n+1} = e$; (**2**) equating the limits from the Root and Ratio tests, results in $0 \leq \lim \frac{n}{b_n} = \lim \left(\frac{n^n}{(2n-1)!!L_n}\right)^{1/n} = \lim \frac{(n+1)^{n+1}}{(2n+1)!!L_{n+1}} \frac{(2n-1)!!L_n}{n^n} = \lim \frac{n+1}{2n+1}(1+\frac{1}{n})^n \frac{L_n}{L_{n+1}} = \frac{e}{1+\sqrt{5}} < 1$; (**3**) let $c_n = \frac{b_{n+1}}{b_n}$ so that $b_{n+1}-b_n = \frac{b_n}{n}(c_n-1)n = \frac{b_n}{n}\left(\frac{c_n-1}{\log c_n}\right)\log c_n^n$; (**4**) $\lim c_n = \lim \frac{b_{n+1}}{n+1}\frac{n}{b_n}\frac{n+1}{n} = 1$ from (**2**) above, hence based on $\lim_{x\to 1}\frac{x-1}{\log x} = \lim_{x\to 1}\frac{1}{\frac{1}{x}} = 1$ (L'Höpital's) we've $\lim_{n\to\infty}\frac{c_n-1}{\log c_n} = 1$. If $d_n = (2n-1)!!L_n$ then $\lim c_n^n = \lim \frac{b_{n+1}^{n+1}}{b_n^n b_{n+1}} = \lim \frac{d_{n+1}}{nd_n}\frac{n+1}{b_{n+1}}\frac{n}{n+1} = \lim \frac{(2n+1)!!L_{n+1}}{n(2n-1)!!L_n}\lim\frac{n+1}{b_{n+1}} = \lim \frac{(2n+1)}{n}\frac{L_{n+1}}{L_n}\lim\frac{n+1}{b_{n+1}} = (1+\sqrt{5})\frac{e}{1+\sqrt{5}} = e$, from (**2**) above. By continuity, $\lim \log c_n^n = 1$ which implies $\lim(b_{n+1}-b_n) = \frac{1+\sqrt{5}}{e}$; (**5**) let $g_n = \frac{a_{n+1}}{a_n}$ and rewrite $(a_{n+1}-a_n)n = a_n\frac{g_n-1}{\log g_n}\log g_n^n$. Using $\lim a_n = e$ and $\lim g_n = 1$, we argue as in (**3**) above to get $\lim \frac{g_n-1}{\log g_n} = 1$. Clearly, $\log g_n^n = \frac{n}{n+1}\frac{\log(1+\frac{1}{n+1})}{\frac{1}{n+1}} - \frac{n^2}{(n+1)^2}\frac{\log(1-\frac{1}{(n+1)^2})}{\frac{-1}{(n+1)^2}}$ and hence $\lim g_n^n = 0$ due to $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$ (L'Höpital's). We conclude $\lim(a_{n+1}-a_n)n = 0$. Combining all these evaluations, the required limit becomes

$$\lim_{n\to\infty}(f_{n+1}-f_n) = \lim_{n\to\infty}a_{n+1}\cdot\lim_{n\to\infty}(b_{n+1}-b_n) + \lim_{n\to\infty}\frac{b_n}{n}\cdot\lim_{n\to\infty}(a_{n+1}-a_n)n = 1+\sqrt{5}. \qquad \square$$

Typeset by $\mathcal{AMS}$-TEX