SOLUTION TO PROBLEM #11876
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Problem #11876. *Proposed by A. Cibulis, Latvia.* Let a, b be the roots of $x^2 + x + \frac{1}{2} = 0$. Find

$$\sum_{n=1}^{\infty} \frac{(-1)^n (a^n + b^n)}{n+2}.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Notice that $ab = \frac{1}{2}$ and $a+b = -1$. In fact, let $a = \frac{-1+i}{2}$ and $b = \frac{-1-i}{2}$ where $i = \sqrt{-1}$. Since $|a| = |b| = \frac{1}{\sqrt{2}}$. The series expansions $\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$ and $\sum_{n\geq 0} b^n z^n = \frac{1}{1-bz}$ are valid for $|z| < \sqrt{2}$. In this domain, we may add and safely integrate (using path-independence of analytic functions)

$$\begin{aligned}
\sum_{n\geq 0} \frac{(a^n + b^n)(-1)^{n+2}}{n+2} &= \int_0^{-1} \sum_{n\geq 0} (a^n + b^n) z^{n+1} dz = \int_0^{-1} \left(\frac{z}{1-az} + \frac{z}{1-bz} \right) dz \\
&= \int_0^{-1} \left(2z - \frac{(a+b)z^2}{abz^2 - (a+b)z + 1} \right) dz = \int_0^{-1} \left(\frac{2z + z^2}{\frac{1}{2}z^2 + z + 1} \right) dz \\
&= \left[2z - 4\arctan(z+1) \right]_0^{-1} = -2 + \pi.
\end{aligned}$$

That means, the required sum is $\sum_{n\geq 1} \frac{(-1)^n (a^n + b^n)}{n+2} = -1 + \sum_{n\geq 0} \frac{(-1)^n (a^n + b^n)}{n+2} = \pi - 3.$ $\square$

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