SOLUTION TO PROBLEM #11883 OF THE AMERICAN MATHEMATICAL MONTHLY

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Problem #11883. *Proposed by H. Ohtsuka, Saitama, Japan.* For $|q| > 1$, prove that

$$\sum_{k=0}^{\infty} \frac{1}{(q^{2^0}+q)(q^{2^1}+q)\cdots(q^{2^k}+q)} = \frac{1}{q-1}\prod_{i=0}^{\infty}\frac{1}{q^{1-2^i}+1}.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Replacing $q \to \frac{1}{q}$, the assertion is

$$\sum_{k=0}^{\infty}\prod_{i=0}^{k}\frac{q^{2^i}}{1+q^{2^i-1}} = \frac{q}{1-q}\prod_{i=0}^{\infty}\frac{1}{1+q^{2^i-1}}. \tag{1}$$

By induction on $N \in \mathbb{Z}^{\geq 0}$, we show that

$$\sum_{k=0}^{N}\prod_{i=0}^{k}\frac{q^{2^i}}{1+q^{2^i-1}} = \frac{q-q^{2^{N+1}}}{1-q}\prod_{i=0}^{N}\frac{1}{1+q^{2^i-1}}. \tag{2}$$

For $N=0$, both sides equal $\frac{q}{2}$. Assume (2) for N. The case $N+1$ is rather routine from here:

$$\begin{aligned}
\sum_{k=0}^{N+1}\prod_{i=0}^{k}\frac{q^{2^i}}{1+q^{2^i-1}} &= \frac{q-q^{2^{N+1}}}{1-q}\prod_{i=0}^{N}\frac{1}{1+q^{2^i-1}} + \frac{q^{2^{N+2}-1}}{1+q^{2^{N+1}-1}}\prod_{i=0}^{N}\frac{1}{1+q^{2^i-1}} \\
&= \frac{1}{1-q}\left[(q-q^{2^{N+1}})(1+q^{2^{N+1}-1}) + (1-q)q^{2^{N+2}-1}\right]\prod_{i=0}^{N+1}\frac{1}{1+q^{2^i-1}} \\
&= \frac{q-q^{2^{N+2}}}{1-q}\prod_{i=0}^{N}\frac{1}{1+q^{2^i-1}}.
\end{aligned}$$

Finally, the limit $N \to \infty$ in equation (2) implies (1), since $|q| < 1$ and hence $q^{2^{N+1}} \to 0$. $\square$

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