SOLUTION TO PROBLEM #11884
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Problem #11884. *Proposed by C. Lupu, University of Pittsburgh, Pittsburgh, PA and T. Lupu, Decebal High School, Constanta, Romania.* Let f be a real-vlaued function on $[0,1]$ such that f and its first two derivatives are continuous. Prove that if $f(\frac{1}{2})=0$ then

$$\int_0^1 (f''(x))^2dx \geq 320\left(\int_0^1 f(x)dx\right)^2.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Apply integration by parts: once in the form of $u=x-x^2, v'=f''$ and then followed by in the form of $u=1-2x, v'=f'$ to gather $\int_0^1(x-x^2)f''(x)dx = f(1)+f(0)-2\int_0^1 f(x)dx$. On the other hand, from the Euler-Maclaurin Formula (applied to the function $h(x)=f(\frac{x}{2})$ over $[0,2]$), we obtain

$$f(0)+f(1/2)=2\int_0^1 f(x)dx - \frac{f(1)-f(0)}{2}+\frac{f'(1)-f'(0)}{24}-\frac{1}{4}\int_0^1 B_2(\{2x\})f''(x)dx;$$

where $\{y\}$ denotes the fractional part of $y\in\mathbb{R}$ and $B_2(y)=y^2-y+\frac{1}{6}$ is the 2^{nd} Bernoulli polynomial. Using $\int_0^1 f''dx=f'(1)-f'(0)$, the assumption $f(\frac{1}{2})=0$ and above-noted identity, we arrive at

$$\begin{aligned}\int_0^1 fdx &= \frac{f(1)+f(0)}{4}+\frac{1}{8}\int_0^1\left(B_2(\{2x\})-\frac{1}{6}\right)f''(x)dx\\ &=\frac{1}{2}\int_0^1 fdx+\frac{1}{4}\int_0^1(x-x^2)f''dx++\frac{1}{8}\int_0^1\left(B_2(\{2x\})-\frac{1}{6}\right)f''(x)dx.\end{aligned}$$

That means, $\int_0^1 fdx = \frac{1}{4}\int_0^1\left(\{2x\}^2-\{2x\}+2x-2x^2\right)f''(x)dx$. By Cauchy-Schwartz inequality, we have $\left(\int_0^1 fdx\right)^2 \leq \frac{1}{16}\int_0^1\left(\{2x\}^2-\{2x\}+2x-2x^2\right)^2dx\cdot\int_0^1(f'')^2dx = \frac{1}{320}\int_0^1(f'')^2dx$ since

$$\begin{aligned}\int_0^1\left(\{2x\}^2-\{2x\}+2x-2x^2\right)^2dx &= \frac{1}{2}\int_0^2\left(\{u\}^2-\{u\}+u-\frac{1}{2}u^2\right)^2du\\ &=\frac{1}{8}\int_0^1 u^4du+\frac{1}{8}\int_1^2(u-2)^4du=\frac{1}{20}.\qquad\square\end{aligned}$$

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