SOLUTION TO PROBLEM #11885

Problem #11885. *Proposed by Cornel Ioan Valean, Teremia Mare, Romania.* Prove that

$$\sum_{p=1}^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(m+n)^4+(m+n)^2(m+p)^2}=\frac{3}{2}\zeta(3)-\frac{5}{4}\zeta(4).$$

Here ζ denotes the Riemann zeta function.

Solution by Tewodros Amdeberhan and Victor H. Moll, Tulane University, LA, USA. For $k,\ell\in\mathbb{N}$, if $f(k,\ell):=\#\{(m,n,p)\in\mathbb{N}^3 | m+n=k, n+p=\ell\}$ then it easy to see that

$$f(k,\ell)=\begin{cases}\ell-1, & \text{when } 0\le\ell\le k\\ k-1, & \text{when } \ell\ge k+1.\end{cases}$$

The required triple sum equals $\sum_{k\ge1}\sum_{\ell=1}^{k}\frac{\ell-1}{k^2(k^2+\ell^2)}+\sum_{k\ge1}\sum_{\ell=k+1}^{\infty}\frac{k-1}{k^2(k^2+\ell^2)}=S_1+S_2+S_3$ where

$$S_1:=\sum_{k\ge1}\sum_{\ell=1}^{k}\frac{\ell}{k^2(k^2+\ell^2)}=\sum_{k\ge1}\sum_{\ell=1}^{k}\frac{1}{k^2\ell}-\sum_{k\ge1}\sum_{\ell=1}^{k}\frac{1}{\ell(k^2+\ell^2)}=2\zeta(3)-\sum_{k\ge1}\sum_{\ell=1}^{k}\frac{1}{\ell(k^2+\ell^2)}$$

by partial fractions and a result of Euler $\sum_{k\ge1}\frac{H_k}{k^2}=2\zeta(3)$. Here $H_k=\sum_{\ell=1}^{k}\frac{1}{\ell}$. Furthermore,

$$S_2:=\sum_{k\ge1}\sum_{\ell\ge k+1}\frac{1}{k(k^2+\ell^2)}=\sum_{\ell\ge1}\sum_{k=1}^{\ell}\frac{1}{k(k^2+\ell^2)}-\frac{1}{2}\zeta(3)=\sum_{k\ge1}\sum_{\ell=1}^{k}\frac{1}{\ell(k^2+\ell^2)}-\frac{1}{2}\zeta(3),$$

$$S_3:=\sum_{k,\ell\ge1}\frac{1}{k^2(k^2+\ell^2)}=\frac{1}{2}\sum_{k,\ell\ge1}\frac{1}{k^2(k^2+\ell^2)}+\frac{1}{2}\sum_{k,\ell\ge1}\frac{1}{\ell^2(k^2+\ell^2)}=\frac{1}{2}\sum_{k,\ell\ge1}\frac{\frac{1}{k^2}+\frac{1}{\ell^2}}{k^2+\ell^2}=\frac{1}{2}\zeta(2)^2.$$

Combining the above, we arrive at $S_1+S_2+S_3=\frac{3}{2}\zeta(3)-\frac{1}{2}\zeta(2)^2=\frac{3}{2}\zeta(3)-\frac{5}{4}\zeta(4)$. $\square$

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