SOLUTION TO PROBLEM #11886
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Problem #11886. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Suppose $n \geq 3$, and let $y_1, \ldots, y_n$ be a list of real numbers such that $2y_{k+1} \leq y_k + y_{k+2}$ for $1 \leq k \leq n-2$. Suppose further that $\sum_{k=1}^n y_k = 0$. Prove that

$$\sum_{k=1}^n k^2 y_k \geq (n+1) \sum_{k=1}^n k y_k,$$

and determine when equality holds.

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Assuming n is even, the assertion is equivalent to $\sum_{k=1}^n k(n+1-k)y_k = \sum_{k=1}^{n_*} k(n+1-k)x_k \leq 0$ where $x_k = y_k + y_{n+1-k}$ and $n_* = \lfloor \frac{n}{2} \rfloor$. We are given **(a)**: $\sum_{k=1}^{n_*} x_k = 0$. The inequalities **(b)**: $2y_{k+1} \leq y_k + y_{k+2}$ imply $y_{k+1} - y_k \leq y_{k+2} - y_{k+1}$ arranged in order $1 \leq k \leq n-2$. Now, add the two central inequalities and then the four central inequalities and so on. This process yields the chain **(c)**: $x_{n_*} \leq \cdots \leq x_2 \leq x_1$. Due to **(a)**, it is clear that $x_{n_*} \leq 0$ and $x_1 \geq 0$. Denote $\boldsymbol{j} := max\{k : x_k \geq 0\}$ to proceed as follows. Using **(a)**, for each $k = 1, 2, \ldots, \boldsymbol{j}$, in that order, make a successive substitution of $k(n+1-k)x_k$ by $-k(n+1-k)\left(\sum_{m=1}^{k-1} x_m + \sum_{m=k+1}^{n_*} x_m\right)$ into $\sum_{k=1}^{n_*} k(n+1-k)x_k$. After collecting terms together, observe that the relative difference in the coefficients remains the same with $x_{\boldsymbol{j}}$ being absent! Hence, we've **(d)**: $\sum_{k=1}^{n_*} k(n+1-k)x_k = -\sum_{k=1}^{\boldsymbol{j}} b_k x_k + \sum_{k=\boldsymbol{j}+1}^{n_*} b_k x_k$ where (i) $x_k \geq 0, b_k \geq 0$ if $k < \boldsymbol{j}$; (ii) $x_k \leq 0, b_k \geq 0$ if $k > \boldsymbol{j}$. Therefore $\sum_{k=1}^{n_*} k(n+1-k)x_k \leq 0$.
In **(d)**, if $\sum_{k=1}^{n_*} k(n+1-k)x_k = 0$ then (i), (ii) and **(a)** imply that $y_k + y_{n+1-k} = x_k = 0$ for all k. This fact, together with a comparison of dual indices in **(b)**, shows that $2y_{k+1} \leq y_k + y_{k+2}$ and $-2y_{k+1} \leq -y_k - y_{k+2}$ (the latter is from $2y_{n-k} \leq y_{n+1-k} + y_{n-k-1}$). So, $2y_{k+1} = y_k + y_{k+2}$ for $1 \leq k \leq n_* - 1$. Beginning with $2y_{n_*} = y_{n_*-1} + y_{n_*+1} = y_{n_*-1} - y_{n_*}$, we get successively $y_{n_*-1} = 3y_{n_*}$ and then $y_{n_*-2} = 5 * y_{n_*}$, $y_{n_*-3} = 7 * y_{n_*}$ etc. That means equality holds in **(d)** iff

$$[y_1, y_2, \ldots, y_n] = [n-1, n-3, \ldots, 5, 3, 1, -1, -3, -5, \ldots, -(n-3), -(n-1)]y_{n_*}.$$

A similar argument proves the case n is odd, i.e. $\sum_{k=1}^n k(n+1-k)y_k \leq 0$ and equality holds iff

$$[y_1, y_2, \ldots, y_n] = [n_*, n_* - 1, \ldots, 3, 2, 1, 0, -1, -2, -3, \ldots, -(n_* - 1), -n_*]y_{n_*}.$$

Note. The above method of proof is elementary, constructive and doesn't rely on extra tools. □