SOLUTION TO PROBLEM #11890
OF THE AMERICAN MATHEMATICAL MONTHLY

Problem #11890. *Proposed by G. Apostolopoulos, Greece.* Find all $x \in (1, +\infty)$ such that

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{1}{x^{2k+1}} + \left(\frac{x-1}{x+1} \right)^{2k+1} \right) = \frac{1}{2} \int_0^x \frac{dx}{\sqrt{1+t^2}}.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Standard methods evaluate the two infinite sums (for $x > 1$) and the integral to yield the equation

$$\frac{1}{2} \log \left(\frac{x+1}{x-1} \right) + \frac{1}{2} \log x = \frac{1}{2} \log(x + \sqrt{1+x^2}).$$

After combining the logarithms and simplification, we are lead to solve $x^4 - 2x^3 - 2x^2 - 2x + 1 = 0$. This palindrome polynomial hints at its roots: $g \pm \sqrt{g}, -\frac{1}{g} \pm i\sqrt{\frac{1}{g}}$ where $i = \sqrt{-1}$ and $g = \frac{1+\sqrt{5}}{2}$. In fact, it is easy to check that $x^4 - 2x^3 - 2x^2 - 2x + 1 = (x^2 - 2gx + 1)(x^2 + \frac{2}{g}x + 1) = 0$. Therefore, there is a unique real number $x = g + \sqrt{g} > 1$ satisfying the equality in the problem. $\square$