SOLUTION TO PROBLEM #11892
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Problem #11892. *Proposed by F. Perdomo and A. Plaza, Spain.* Let f be a real-valued continuously differentiable function on $[a, b]$ with positive derivative on (a, b). Prove that, for all pairs (x_1, x_2) with $a \leq x_1 < x_2 \leq b$ and $f(x_1)f(x_2) > 0$, there exists $t \in (x_1, x_2)$ such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = t - \frac{f(t)}{f'(t)}.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Without loss of generality, assume $A := f(a) < f(b) := B$. Since $f' > 0$, it follows that $y = f(x)$ has an inverse $x = g(y)$ for $y \in (A, B)$. Since $AB > 0$, it is no loss to suppose $0 < A < B$. Let $y_1 = f(x_1), y_2 = f(x_2)$ or $x_1 = g(y_1), x_2 = g(y_2)$. The rest is a consequence of Pompeiu's Mean Value Theorem: there exists $\xi \in (y_1, y_2)$ such that

$$\frac{y_1 g(y_2) - y_2 g(y_1)}{y_1 - y_2} = g(\xi) - \xi g'(\xi).$$

Let $t = g(\xi)$. Then $\xi = f(t), g'(\xi) = \frac{1}{f(g(\xi)} = \frac{1}{f'(t)}$ and the desired result

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_1) - f(x_2)} = t - \frac{f(t)}{f'(t)}. \qquad \square$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX