SOLUTION TO PROBLEM #11897
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Problem #11897. *Proposed by P. Dalyay, Hungary.* Prove for $n \geq 0$,

$$\sum_{\substack{k+j=n \\ k,j\geq 0}} \frac{1}{k+1}\binom{2k}{k}\binom{2j+2}{j+1} = 2\binom{2n+2}{n}.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. The identity is equivalent to $\sum_{k=0}^{n} \frac{1}{k+1}\binom{2k}{k}\binom{2n-2k+2}{n-k+1} = 2\binom{2n+2}{2}$, or $\sum_{k=0}^{n+1} \frac{1}{k+1}\binom{2k}{k}\binom{2n-2k+2}{n-k+1} = 2\binom{2n+2}{n} + \frac{1}{n+2}\binom{2n+2}{n+1} = \binom{2n+3}{n+1}$. Recalling the well-known generating function $\sum_{k\geq 0}\binom{2k}{k}x^k = \frac{1}{\sqrt{1-4x}}$ and $\sum_{k\geq 0}\binom{2k}{k}\frac{x^k}{k+1} = \frac{2}{1+\sqrt{1-4x}}$, the Cauchy product formula offers $\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k}\right)x^n = \frac{2}{\sqrt{1-4x}(1+\sqrt{1-4x})}$. Also that, $\frac{2}{\sqrt{1-4x}(1+\sqrt{1-4x})} = \frac{2}{\sqrt{1-4x}} - \frac{2}{1+\sqrt{1-4x}} = \sum_{n\geq 0}\left(2\binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n}\right)x^n = \sum_{n\geq 0}\binom{2n+1}{n}x^n$. Comparing coefficients leads to the desired identity $\sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} = \binom{2n+1}{n}$. $\square$

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