SOLUTION TO PROBLEM #11899
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Problem #11899. Proposed by J. Sorel, Romania. Show that for any positive integer n,

$$\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}+\sum_{k=n+1}^{2n+1}\binom{2n}{k-1}\binom{2n+1}{k}=\binom{4n+1}{2n}+\binom{2n}{n}^2.$$

Solution by Tewodros Amdeberhan, Tulane University, LA, USA. Start with $A_1 := \sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}$, $A_2 := \sum_{k=n+1}^{2n+1}\binom{2n}{k}\binom{2n+1}{k}$, $B_1 := \sum_{k=n+1}^{2n+1}\binom{2n}{k-1}\binom{2n+1}{k}$ and $B_2 := \sum_{k=0}^{n}\binom{2n}{k-1}\binom{2n+1}{k}$. Re-indexing gives $A_1 = B_1, A_2 = B_2$. The required identity is $A_1 + B_1 = 2A_1 = \binom{4n+1}{2n}+\binom{2n}{n}^2$. In view of the Vandermonde-Chu identity $A_1 + A_2 = \binom{4n+1}{2n}$, it suffices to prove that $A_1 - A_2 = A_1 - B_2 = \binom{2n}{n}^2$. That is, $\sum_{k=0}^{n}\binom{2n+1}{k}\left[\binom{2n}{k}-\binom{2n}{k-1}\right]=\sum_{k=0}^{n}\binom{2n+1}{n-k}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]=\binom{2n}{n}^2$.

It is routine to check that $\binom{2n+1}{n-k}\left[\binom{2n}{n-k}-\binom{2n}{n-k-1}\right]=\binom{2n+1}{n-k}^2\frac{2k+1}{2n+1}=G(n,k)-G(n,k+1)$ where $G(n,k)=\binom{2n}{n+k}^2$. But, $\sum_{k=0}^{n}[G(n,k)-G(n,k+1)]=G(n,0)-G(n,n+1)=G(n,0)=\binom{2n}{n}^2$. $\square$

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