SOLUTION TO PROBLEM #11901

Problem #11901. Proposed by Donald Knuth, Stanford University, Stanford, CA. For $n \in \mathbb{Z}^+$, let $[n] = \{1, \ldots, n\}$. Define the functions $\uparrow$ and $\downarrow$ on $[n]$ by $\uparrow x = \min\{x+1, n\}$ and $\downarrow x = \min\{x-1, n\}$. How many distinct mappings from $[n]$ to $[n]$ occur as compositions of $\uparrow$ and $\downarrow$?

Solution by Tewodros Amdeberhan, Tulane University, LA, and Richard P. Stanley, Massachusetts Institute of Technology, MA, USA. Each of the desired mappings f may be viewed as an n-letter word $w = w_1 \cdots w_n$ with alphabets in $[n]$. Compositions of $\uparrow, \downarrow$ are **completely characterized** by:

(a) the w_i's are weakly increasing;

(a) only w_1 and w_n can be repeated,

i.e., $w = w_1^n$ or $w = w_1^k w_1(w_1+1)\cdots(w_1+j)(w_1+j)^\ell$ for some $j \geq 1, k, \ell \geq 0$ with $k+j+1+\ell = n$. There are n possible ways for the case $w = w_1^n$. There are $n-j$ possible j consecutive letters $w_1(w_1+1)\cdots(w_1+j)$, each can be placed in $n-j$ possible positions to form part of a word w. If $j = n-1$ then $w = 12\cdots n$. This is impossible because w can never be one-to-one. Therefore, the total number of mappings of the required type equals

$$n + \sum_{j=1}^{n-2}(n-j)^2 = n - 1 + \sum_{k=1}^{n-1} k^2 = n - 1 + \frac{1}{4}\binom{2n}{3}$$

whenever $n \geq 2$. The special case $n = 1$ clearly gives exactly one mapping. $\square$

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